Table of Contents – Unit 1 – Characteristics of Functions

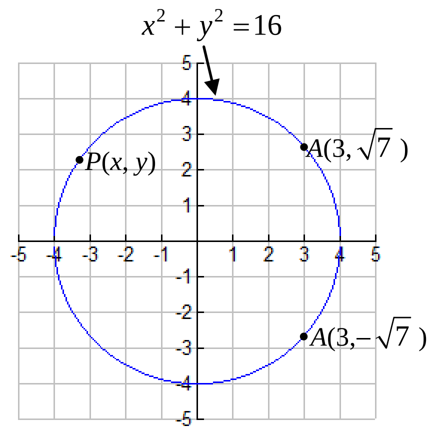
Introduction to Functions

Relations and Functions

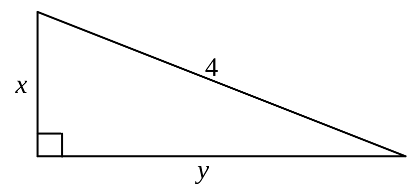
Equations called ***formulas*** express mathematical ***relationships***.

e.g. $x^2 + y^2 = 16$

- The equation $x^2 + y^2 = 16$ expresses a ***relationship*** between *x* and *y*.
- The equation $x^2 + y^2 = 16$ can describe the set of points *P*(*x*, *y*) lying on a circle of radius 4 with centre (0, 0).
- The equation $x^2 + y^2 = 16$ can also describe ***every right triangle*** with a ***hypotenuse*** of length 4.
- By solving $x^2 + y^2 = 16$ for *y*, we obtain two "answers," $y = \sqrt{16 - x^2}$ and $y = -\sqrt{16 - x^2}$. As we can see from the graph and the accompanying table of values, ***for any given value of x***, where $-4 < x < 4$, there are ***two possible values*** of *y*.



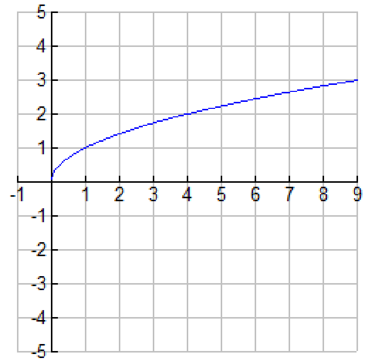
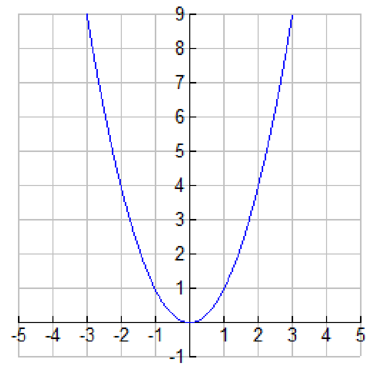
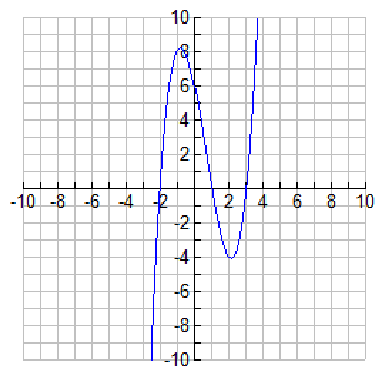
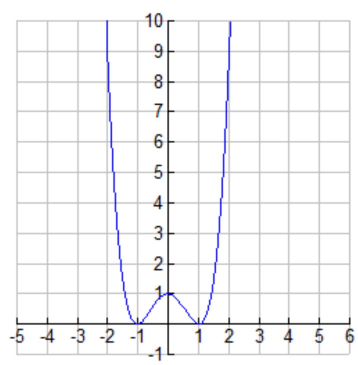
x	sqrt(16-x^2)	-sqrt(16-x^2)
-4.5	undef	undef
-4	0	0
-3.5	1.93649	-1.93649
-3	2.64575	-2.64575
-2.5	3.1225	-3.1225
-2	3.4641	-3.4641
-1.5	3.7081	-3.7081
-1	3.87298	-3.87298
-0.5	3.96863	-3.96863
0	4	-4
0.5	3.96863	-3.96863
1	3.87298	-3.87298
1.5	3.7081	-3.7081
2	3.4641	-3.4641
2.5	3.1225	-3.1225
3	2.64575	-2.64575
3.5	1.93649	-1.93649
4	0	0
4.5	undef	undef



- Since for certain values of *x*, there are two possible values of *y*, $x^2 + y^2 = 16$ is ***NOT a function***.

- Any equation that expresses a ***relationship between two unknowns*** is called a ***RELATION***.
- If ***for each possible value of x*** there is ***one and only one corresponding value of y***, a relation is called a ***FUNCTION***.
- A ***vertical line*** can intersect the graph of a function ***at one and only one point***.
- You have become accustomed to the convention of writing "*x*" for the ***independent variable*** and "*y*" for the ***dependent variable***. To indicate that a relation is a function, we use the notation "$f(x)$" instead of "*y*" to denote the value of the ***dependent variable***. This is called ***function notation***.
- This notation also helps to remind us that the ***value of y depends on the value of x***.
- Consider the function *f* defined by the equation $f(x) = x^2$. We read this as "**f of x equals x squared**" and it means that the value of $f(x)$ (i.e. the value of the dependent variable) is obtained by squaring the value of the independent variable *x*. Note that "*f*" is the ***name*** of the function and "$f(x)$" is the *y*-value. Note also that "*f* of *x*" can be seen as a short form for "**f is a function of x**," which means that the function's value **depends on x**.
- If we write, for instance, "$f(2)$," we mean "the *y*-value obtained when *x* = 2." The notation "$f(2)$" is read "*f* at 2."
- Traditionally, the letter "*f*" is used to denote functions because the word "function" begins with the letter "*f*." This does not disqualify other letters! Functions can equally well be given other names such as *g*, *h* and *q*.

Examples of Functions

Equation of Function written without Function Notation	*Equation of Function written with Function Notation*	*Graph*	*Examples*
$y=\sqrt{x}$	$f(x)=\sqrt{x}$		$f(4)=\sqrt{4}=2$ $\therefore$(4, 2) lies on the graph of f $f(25)=\sqrt{25}=5$ $\therefore$(25, 5) lies on the graph of f $f(2)=\sqrt{2}\doteq 1.414$ $\therefore(2, \sqrt{2})$ lies on the graph of f $f(-1)=\sqrt{-1}$, which is undefined in the set of real numbers
$y=x^2$	$g(x)=x^2$		$g(1)=1^2=1$ $\therefore$(1, 1) lies on the graph of g $g(2)=2^2=4$ $\therefore$(2, 4) lies on the graph of g $g(5)=5^2=25$ $\therefore$(5, 25) lies on the graph of g $g(-3)=(-3)^2=9$ $\therefore$(–3, 9) lies on the graph of g
$y=x^3-2x^2-5x+6$	$h(x)=x^3-2x^2-5x+6$		$h(1)=1^3-2(1)^2-5(1)+6=0$ $\therefore$(1, 0) lies on the graph of h $h(2)=2^3-2(2)^2-5(2)+6=-4$ $\therefore$(2, –4) lies on the graph of h $h(5)=5^3-2(5)^2-5(5)+6=56$ $\therefore$(5, 56) lies on the graph of h $h(-1)=(-1)^3-2(-1)^2-5(-1)+6=8$ $\therefore$(–1, 8) lies on the graph of h
$y=x^4-2x^2+1$	$p(x)=x^4-2x^2+1$		$p(1)=1^4-2(1)^2+1=0$ $\therefore$(1, 0) lies on the graph of p $p(2)=2^4-2(2)^2+1=9$ $\therefore$(2, 9) lies on the graph of p $p(0)=0^4-2(0)^2+1=1$ $\therefore$(0, 1) lies on the graph of h $p(-2)=(-2)^4-2(-2)^2+1=9$ $\therefore$(–2, 9) lies on the graph of p

Viewing Relations and Functions from a Variety of Different Perspectives

Set of Ordered Pairs Perspective

- ***Ordered Pair***
  Two numbers written in the form (x, y) form an ***ordered pair***. As the name implies, ***order is important***.
- ***Set***
  A ***set*** is a group or collection of numbers, variables, geometric figures or just about anything else. Sets are written using ***set braces*** "{}." For example, {1, 2, 3} is the set containing the elements 1, 2 and 3. Note that ***order does not matter*** in a set. The sets $\{a, b, c\}$ and $\{c, a, b\}$ are the same set. Repetition does not matter either, so $\{a, b\}$ and $\{a, a, b, b, b\}$ are the same set.

  The main idea of a set is to group objects that have common properties. For example, the symbol "$\mathbb{Q}$" represents the set of ***rational numbers***, the set of all ***fractions***, including negative fractions and zero. Alternatively, we can think of a ***rational number*** as a ***ratio*** of two integers. That is, all rational numbers share the common property that they can be written in the form $\frac{a}{b}$, where a and b are both integers and $b \neq 0$. Formally, this is written $\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}$. Note that the symbol $\mathbb{Z}$ ***denotes the set of integers*** and the symbol "$\in$" means "is an element of."
- ***Relation***
  A ***relation*** is any set of ***ordered pairs***. For example, the set $\{(2,0),(2,1),(2,2),(2,3),(2,4),(2,5),\ldots\}$ is a relation. In this relation, the number "2" is associated with every non-negative integer. This relation can be expressed more precisely as follows: $\{(x, y) : x = 2, y \in \mathbb{Z}, y \geq 0\}$. This is read as, "the set of all ordered pairs (x, y) such that x is equal to 2 and y is any non-negative integer."
- ***Function***
  A ***function*** is a ***relation*** in which for every "x-value," there is one and only one corresponding "y-value." The relation given above is ***not*** a function because for the "x-value" 2, there are an infinite number of corresponding "y-values." However, the relation $\{(0,2),(1,2),(2,2),(3,2),(4,2),(5,2),\ldots\}$ = $\{(x, y) : x \in \mathbb{Z}, x \geq 0, y = 2\}$ ***is*** a function because there is only one possible "y-value" for each "x-value." Formally, this idea is expressed as follows:

  Suppose that R represents a ***relation***, that is, a set of ordered pairs. Suppose further that $a = b$ whenever $(a,c) \in R$ and $(b,c) \in R$ (i.e. a must equal b if (a,c) and (b,c) are both in R). Then R is called a ***function***.

Exercise

Which of the following relations are functions? Explain.

(a) $\{(0,2),(1,2),(2,2),(0,3)\}$

(b) $\{(0,2),(1,2),(2,2),(5,3)\}$

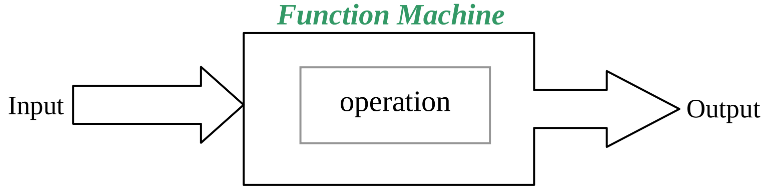
(c) $\{(x, y) : x \in \mathbb{R}, y = x^2\}$

(d) $\{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, x^2 + y^2 = 169\}$

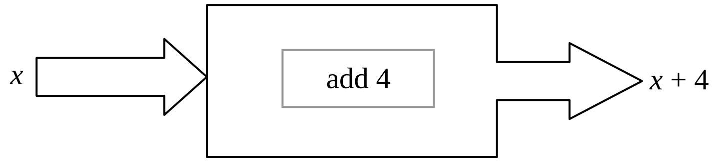
Machine Perspective

Sets of ordered pairs allow us to give very precise mathematical definitions of relations and functions. However, being a quite ***abstract*** concept, the notion of a set may be somewhat difficult to understand. You can also think of a function as a machine that accepts an ***input*** and then produces a ***single output***. ***There is one and only one "output" for every "input."***

Function Machine

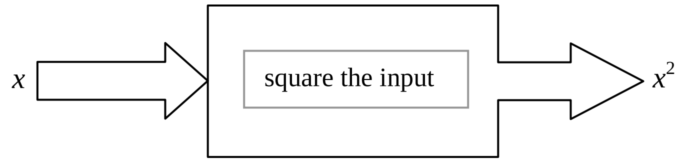


For example, consider the function machine that adds 4 to the input to produce the output.



Exercise

Consider the function machine that takes an input x and outputs x^2.



What is the output if the input is: 4 _______, 0 ________ –4 _______ ?

Is it possible for a specific input to have more than one output? Explain. ______________________________________

____

____

What is the input if the output is 9? ______

Is it possible for a specific output to have more than one input in this case? Explain. _____________________________

____

____

The following ordered pairs show the inputs and outputs of a function machine. What is the operation?

(a) (10, 5), (3, –2), (–1, –6) operation: _____________

(b) (10, 32), (–10, –28), (–3, –7), (1, 5), (0, 2) operation: _____________

Mapping Diagram Perspective

In addition to function machines, functions can be represented using ***mapping diagrams***, ***tables of values***, ***graphs*** and ***equations***. You may not know it but you already have a great deal of experience with functions from previous math courses.

A mapping diagram uses arrows to map each element of the input to its corresponding output value. Remember that a function has only one output for each input.

Can a function have more than one input for a particular output? Explain. _____________________________________

____

____

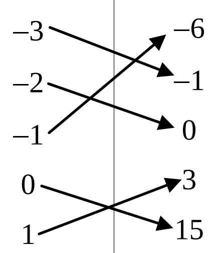
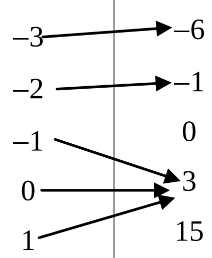
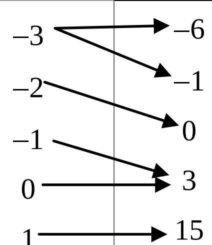
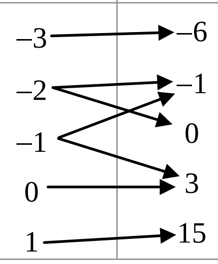
Domain and Range

The set of all possible input values of a function is referred to as the ***domain***. That is, if D represents the domain of a function f, then $D = \{x : (x, y) \in f\}$.

The set of all possible output values of a function is referred to as the ***range***. That is, if R represents the domain of a function f, then $R = \{y : (x, y) \in f\}$.

Exercise 1

In the table, the relations are shown using a mapping diagram. The domain (or inputs) on the left have arrows pointing to the range (or output) on the right. Explain whether the relation is a function or not and justify your answer. The first one is done for you.

Relation as a Mapping Diagram (*Domain* / *Range*)	*Is it a function? Why or why not?*	*Set of ordered pairs*
	Since every element of the domain has only one corresponding element in the range, this relation is a function. This function has a ***one-to-one mapping***.	{(−3, −1), (−2,0), (−1, −6), (0, 15), (1, 3)}
		
		
		

A ***one-to-one mapping*** is explained above. Which relation above has

a ***many-to-one mapping***? ________________________

a ***one-to-many mapping***? ________________________

a ***many-to-many mapping***? ______________________

Exercise 2

Use the Internet to find definitions of the following terms:

one-to-one mapping, one-to-many mapping, many-to-many mapping, surjection, injection, bijection

Numerical Perspective – Tables of Values

Consider the following relations expressed in table form.

(a) Which relations are functions? Justify your answers.
(b) Draw a mapping diagram for each relation.
(c) Write the set of ordered pairs for each relation.

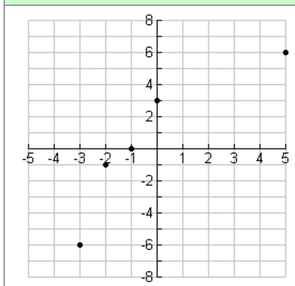
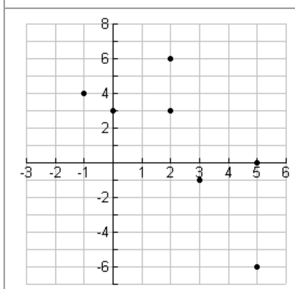
<table>
<tr><th>Relation as a Table of Values</th><th>Is it a function? Why or why not?</th><th>Mapping Diagram</th><th>Set of Ordered Pairs</th></tr>
<tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>–3</td><td>9</td></tr><tr><td>–1</td><td>1</td></tr><tr><td>1</td><td>1</td></tr><tr><td>3</td><td>9</td></tr></table></td><td></td><td></td><td></td></tr>
<tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>–5</td><td>–125</td></tr><tr><td>–1</td><td>–1</td></tr><tr><td>–1</td><td>5</td></tr><tr><td>0</td><td>10</td></tr><tr><td>5</td><td>–125</td></tr></table></td><td></td><td></td><td></td></tr>
</table>

Geometric Perspective – Graphs of Functions

Discrete Relations

A ***discrete relation*** either has a ***finite number*** of ordered pairs ***OR*** the ordered pairs can be ***numbered*** using integers. Graphs of discrete relations consist of either a ***finite*** or an ***infinite*** number of "disconnected" points, much like a "connect-the-dots" picture ***before*** the dots are connected.

Examine the following graphs of ***discrete relations*** and then complete the table. ***DO NOT CONNECT THE DOTS!***

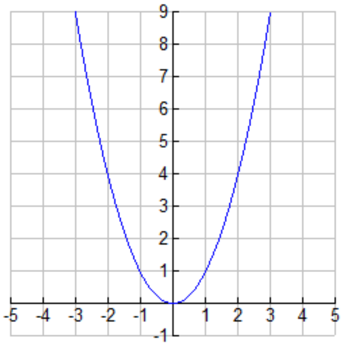
Relation in Graphical Form	*Is it a function? Why or why not?*	*Table of Values*	*Mapping Diagram*	*Set of Ordered Pairs*
				
				

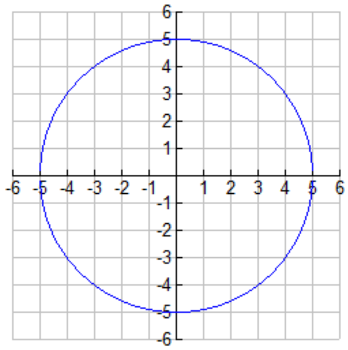
Definition of "Discrete" (from www.dictionary.com)

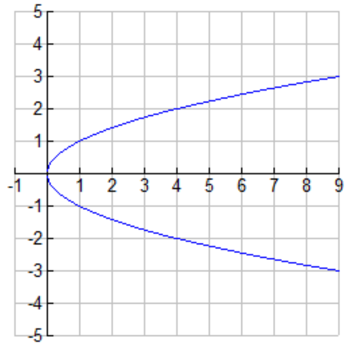
1. apart or detached from others; separate; distinct: *six discrete parts.*
2. consisting of or characterized by distinct or individual parts; discontinuous.
3. ***Mathematics.***
    a. (of a topology or topological space) having the property that every subset is an open set.
    b. defined only for an isolated set of points: *a discrete variable.*
    c. using only arithmetic and algebra; not involving calculus: *discrete methods.*

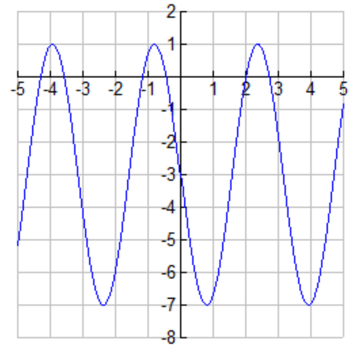
Continuous Relations

Unlike the graphs on the previous page, the following graphs ***do not represent discrete relations***. They are called ***continuous relations*** because their graphs do not consist of disconnected points. (You need to study calculus to learn a more precise definition of continuity. For now, it suffices to think of continuous relations as those whose graphs are "unbroken.")









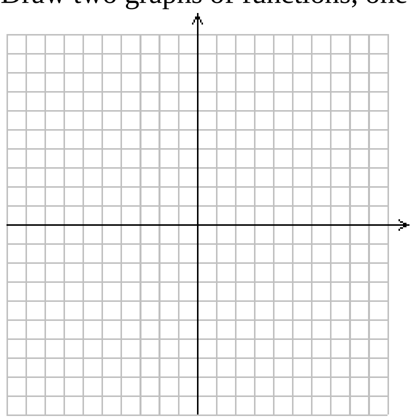
Examine the graphs and decide whether they represent functions.

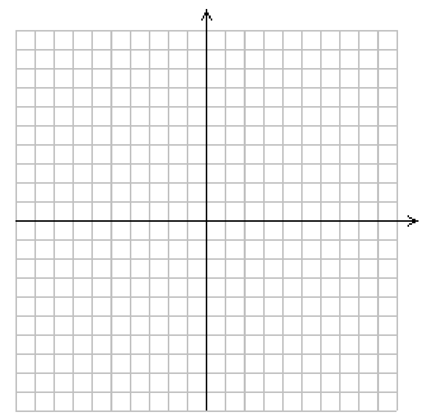
Summary – Vertical Line Test

When you look at a graph, how can you decide whether it represents a function?

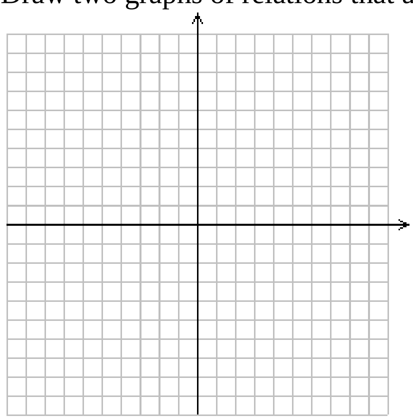
Exercise

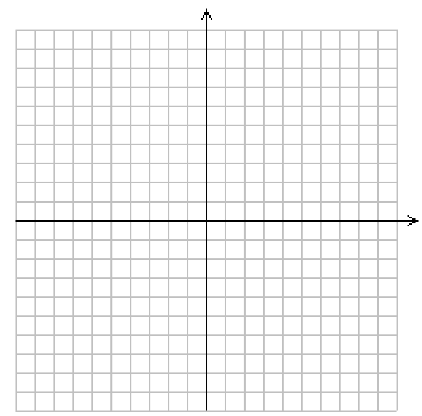
Draw two graphs of functions, one that is discrete and one that is continuous.





Draw two graphs of relations that are ***NOT*** functions, one that is discrete and one that is continuous.



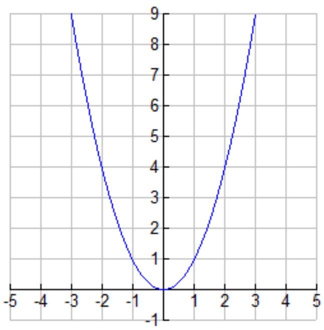
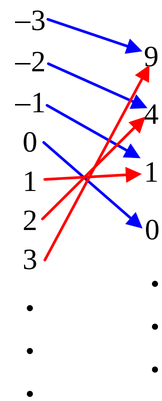
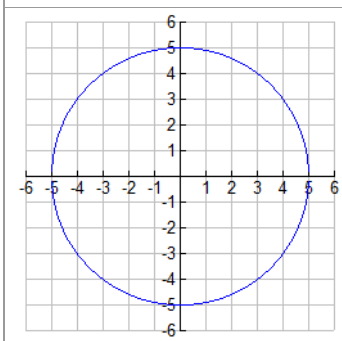
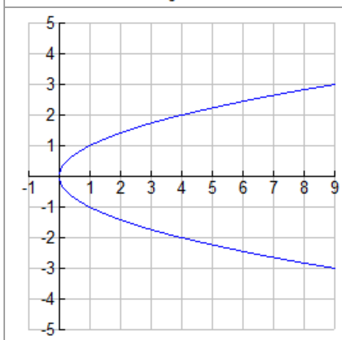
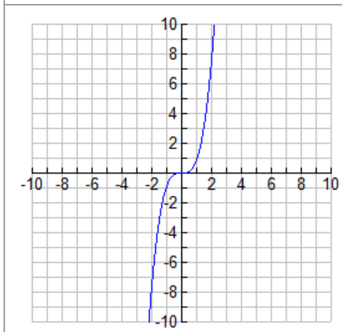


Algebraic Perspective – Equations of Relations

The perspectives given above help us to understand the properties of relations and functions. However, when it comes time to computing with relations and functions, then equations become an indispensable tool!

For each of the following (the first is done for you)

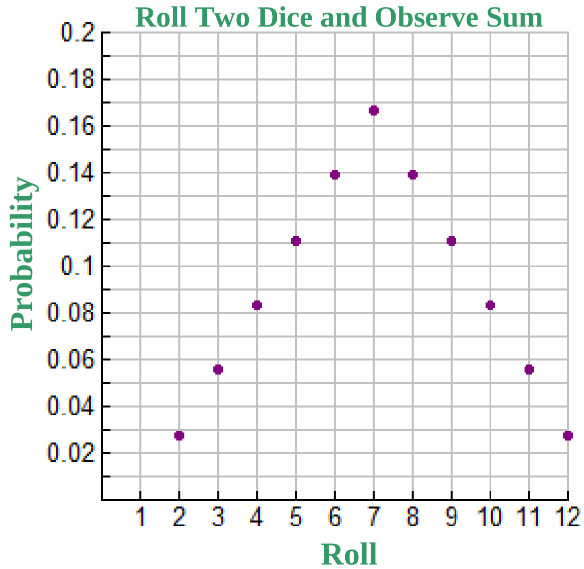
(a) determine whether it is a function.
(b) state an equation that describes the relation.
(c) complete a table of values and a mapping diagram for each relation. Note that the given relations are all ***continuous***, which means that it is impossible to create a complete table of values or mapping diagram. (Why?)
(d) state the domain and range and the type of mapping.

Relation in Graphical Form	*Is it a function? Explain.*	*Equation*	*Table of Values*	*Mapping Diagram*	*Domain and Range (D & R)*	*Type of Mapping*
	This is a function because any vertical line will intersect at only a single point.	$f(x)=x^2$	x | $f(x)$ −2 | 4 −1 | 1 0 | 0 1 | 1 2 | 4 • | • • | • • | •		The domain is the set of all real numbers, that is, $D=\mathbb{R}$. The range is the set of all real numbers greater than or equal to zero, that is, $R=\{y\in\mathbb{R}: y\geq 0\}$	many-to-one
						
						
						

Examples of Discrete and Continuous Relations

(a) *Discrete Function with a Finite Number of Ordered Pairs – Roll Two Dice and Observe the Sum*

Consider rolling two dice and observing the sum. Clearly, there are only eleven possible outcomes, the whole values from 2 to 12 inclusive. It's also clear that some outcomes are more likely than others. If you have played board games that involve dice rolling, you surely have noticed that rolls like "2" and "12" occur infrequently while "7" occurs much more often. The reason for this is that there is only ***one way*** of obtaining "2," for example, but there are ***six ways*** of obtaining "7." By working out all the possibilities, we obtain the probability of each outcome as shown in the table at the right.

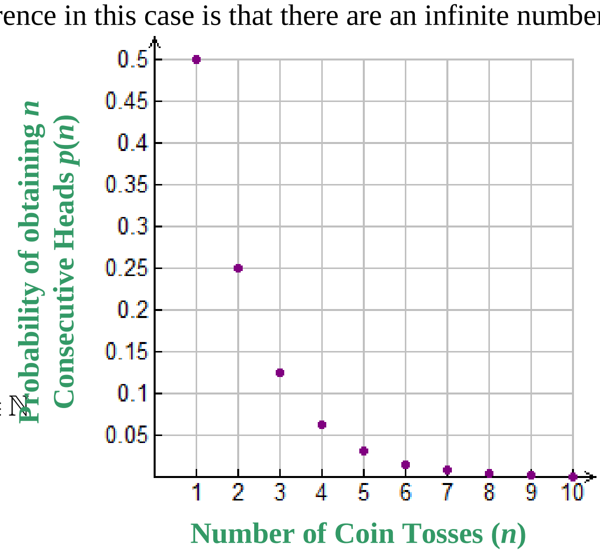


Possible Outcomes	*Probability*
2	$\frac{1}{36} \doteq 0.02778$
3	$\frac{2}{36} \doteq 0.05556$
4	$\frac{3}{36} \doteq 0.08333$
5	$\frac{4}{36} \doteq 0.11111$
6	$\frac{5}{36} \doteq 0.13889$
7	$\frac{6}{36} \doteq 0.16667$
8	$\frac{5}{36} \doteq 0.13889$
9	$\frac{4}{36} \doteq 0.11111$
10	$\frac{3}{36} \doteq 0.08333$
11	$\frac{2}{36} \doteq 0.05556$
12	$\frac{1}{36} \doteq 0.02778$

The main point to remember here is that that we have a ***discrete function***. In this case, there are only a finite number of ordered pairs in this function. Furthermore, they are "disconnected" from each other, as the graph clearly shows. This disconnectedness is a natural consequence of the physical nature of rolling a pair of dice. It is possible to roll a "2" or a "3" but it is ***not possible*** to roll 3.141592654 or any number that is not a whole number between 2 and 12 inclusive!

(b) *Discrete Function with an Infinite Number of Points – Probability of "n" Consecutive "Heads" in "n" Coin Tosses*

Once again, it makes no sense to "connect the dots" in this case. The number of coin tosses ***must be a positive whole number***. It is difficult to imagine how one could make 3.75, 4.99 or $\sqrt{5}$ coin tosses! Once again then, we have a discrete function. The difference in this case is that there are an infinite number of possibilities. There is no limit to the number of coin tosses that one can make. Moreover, there is no bound on the number of consecutive heads that can be obtained. Although it is extremely unlikely to obtain a very large number of consecutive heads, it is ***not*** impossible!



We can take this example one step further and write an equation. Let $p(n)$ represent the probability of obtaining n consecutive heads when a coin is tossed n times. Then,

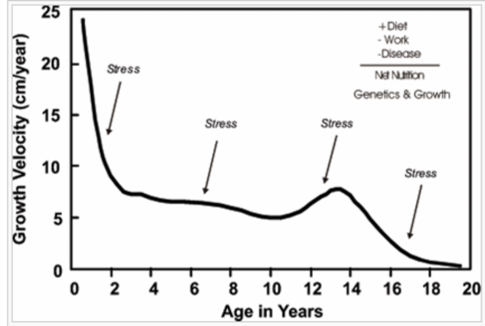
$$p(n) = \left(\frac{1}{2}\right)^n = \frac{1}{2^n} = 2^{-n},\ n \in \mathbb{N}$$

Note that "$n \in \mathbb{N}$" means that n is an element of the set of ***natural numbers***, that is, n is a positive whole number.

Number of Tosses (n)	*Probability of Obtaining n Consecutive Heads*
1	$\frac{1}{2} = 0.5 = 50\%$
2	$\frac{1}{4} = 0.25 = 25\%$
3	$\frac{1}{8} = 0.125 = 12.5\%$
4	$\frac{1}{16} = 0.0625 = 6.25\%$
5	$\frac{1}{32} = 0.03125 = 3.125\%$
• • •	The number of ordered pairs is infinite in this case because (at least in principle) any whole number of consecutive heads is possible.

(c) *Continuous Function*

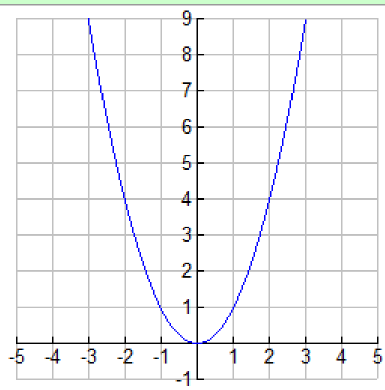
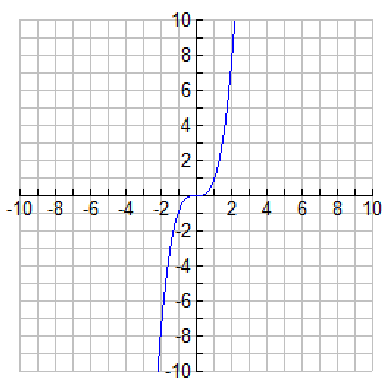
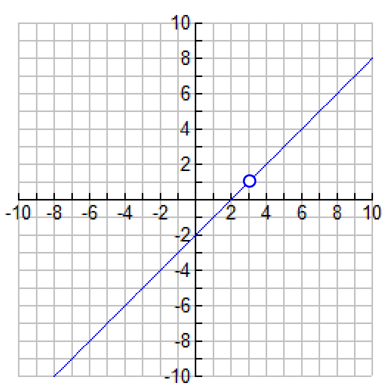
Now consider the ***growth rate*** of humans. At the right is a graph that shows an example of ***growth velocity*** (in cm/year) ***versus age*** (in years). Notice that this function is quite different from the discrete examples in (a) and (b). Unlike the roll of a pair of dice or the number of coin tosses, it is not possible to assign a whole number to a person's age or growth rate. Both quantities vary continuously over a certain range of values. Hence, the graph of a continuous function will not consist of a series of distinct, disconnected points. All the points "fuse" together to form an unbroken curve.



An example of human growth velocity under optimal conditions (Courtesy: Richard Steckel)

Determining Domain and Range given an Equation of a Function

Complete the following table. The first three rows are done for you.

Equation of Function	*Graph*	*Domain and Range*	*Explanation*
$f(x)=x^2$		$D=\mathbb{R}$ $R=\{y\in R: y\geq 0\}$	The domain consists of all real numbers because any real number can be squared. Since the square of any real number must be non-negative, the range consists of all real numbers greater than or equal to zero.
$g(x)=x^3$		$D=\mathbb{R}$ $R=\mathbb{R}$	The domain consists of all real numbers because any real number can be cubed. The range also consists of all real numbers because the cube of a positive real number is positive, the cube of a negative real number is negative and the cube of zero is zero.
$h(x)=\dfrac{x^2-5x+6}{x-3}$		$D=\{x\in\mathbb{R}: x\neq 3\}$ $R=\{y\in\mathbb{R}: y\neq 1\}$	Any value of x is acceptable except for 3. If x = 3, however, the denominator of the expression is zero. Since division by zero is undefined, $x\neq 3$. By noting that $\dfrac{x^2-5x+6}{x-3}=\dfrac{(x-2)(x-3)}{x-3}=x-2$, we see that if x were allowed to equal 3, y would equal 1. Therefore, y can take on any real value except for 1. Note the open circle on the graph of h. This indicates that the ordered pair $(3,1)$ is ***not*** part of the graph. That is, it is ***not included*** in the set of ordered pairs making up the function.

Continued on next page…

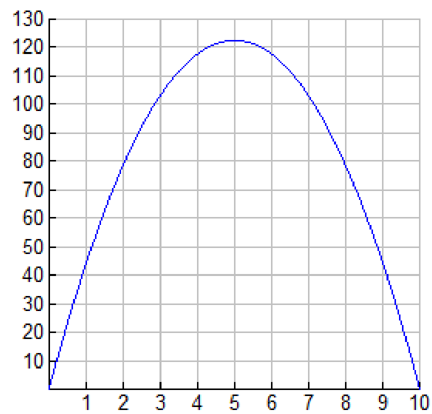
Equation of Function	*Graph*	*Domain and Range*	*Explanation*
$f(x)=x^2+6$			
$g(x)=x^3-2$			
$h(x)=\dfrac{6x^2-x-15}{2x+3}$			
$p(u)=\dfrac{2u-1}{10u^2-7u+1}$			
$f(u)=2\sqrt{u-10}+3$			

Homework

pp. 178 – 179, #1 → 15

Physical Perspective – Applying Functions to a Physical Situation

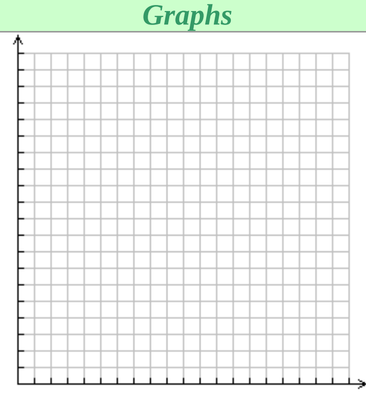
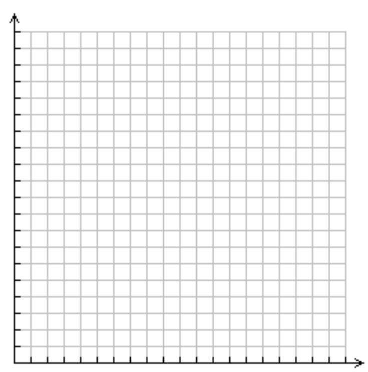
A cannonball is fired vertically into the air with an initial speed of 49 m/s. Its height above the ground in metres, at a time t seconds after it is fired, is given by the function h defined by $h(t)=49t-4.9t^2$. Because this function is used to ***model a physical situation***, we must keep in mind that not all values of t make sense. For example, it is nonsensical to consider negative values of t because the timing begins at t=0 when the cannonball is fired. Similarly, there is no point in considering values of t greater than the time that it takes to fall back to the earth. This is summarized in the table given below.

Physical Situation	*Algebraic Support*	*Graph*	*Domain and Range*
A cannonball is fired ***vertically*** into the air with an initial speed of 49 m/s. Its height above the ground in metres, at a time t seconds after it is fired, is given by the function h defined by $h(t)=49t-4.9t^2$.	$h(t)=49t-4.9t^2$ $=-4.9t^2+49t$ $=-4.9(t^2+10t)$ $=-4.9(t^2+10t+5^2-5^2)$ $=-4.9(t+5)^2+4.9(5)^2$ $=-4.9(t+5)^2+122.5$ $49t-4.9t^2=0$ $\therefore 4.9t(10-t)=0$ $\therefore t=0$ or $10-t=0$ $\therefore t=0$ or $t=10$		$D=\{x\in\mathbb{R}:0\le x\le 10\}$ $R=\{y\in\mathbb{R}:0\le y\le 122.5\}$

Since the function $h(t)=49t-4.9t^2$ is used to model a physical situation, its domain and range are ***restricted***.

Exercise

Complete the following table.

Physical Situation	*Algebraic Support*	*Graphs*	*Domain and Range*
A baseball is hit from a point 1 m above the ground, at an angle of 37º to the ground and with an initial velocity of 40 m/s. (Originally the initial speed given was 100 m/s. This produced very unrealistic answers.) The ***horizontal distance*** travelled by the ball is given by the function $d(t)=(40\cos 37°)t$ and the ***vertical distance*** travelled by the ball is given by the function $h(t)=-4.9t^2+(40\sin 37°)t+1$. How far did the ball travel? What was the maximum height reached by the ball? ***Note that time is measured in seconds and distance is measured in metres.***		 	

Homework

pp. 180 – 181, #16, 19, 20, 22, 23, 24, 25, 26, 28, 29, 31, 32

Transformations of the Base Functions

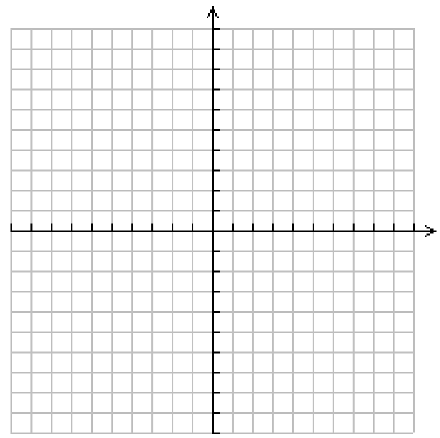
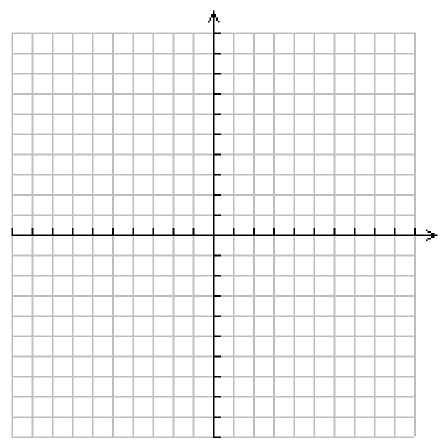
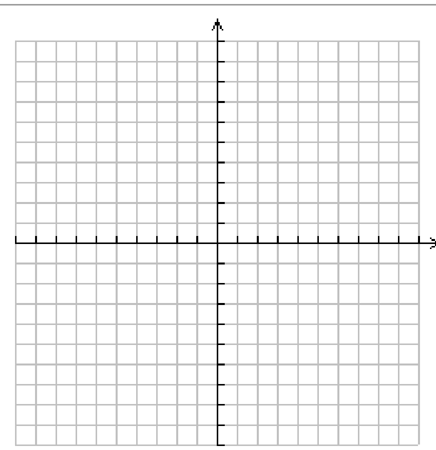
The Base Functions

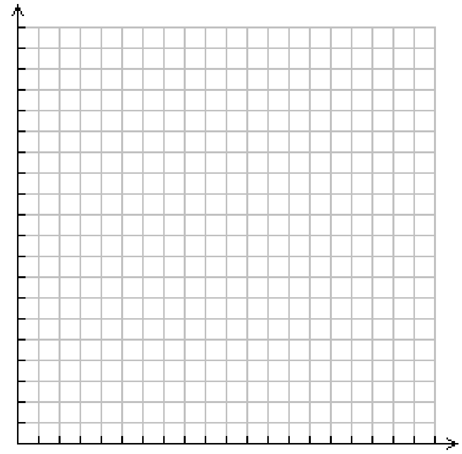
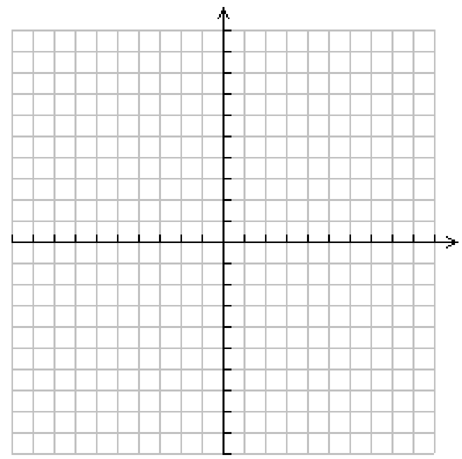
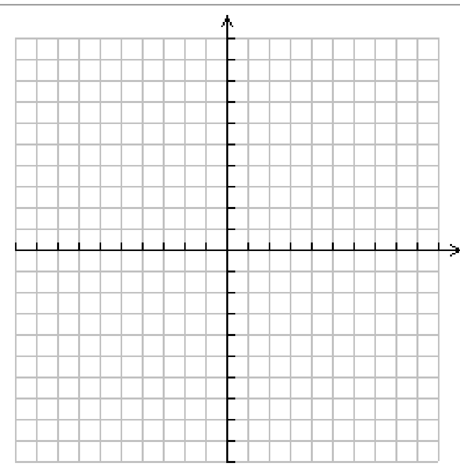
Mathematicians study mathematical objects such as functions to learn about their properties. An important objective of such research is to be able to describe the properties of the objects of study as ***concisely*** as possible. By keeping the basic set of principles as small as possible, this approach greatly simplifies the daunting task of understanding the behaviour of mathematical structures.

Take, for example, the goal of understanding all quadratic functions. Instead of attempting to understand such functions all in one fell swoop, mathematicians divide the process into two simpler steps as shown below.

1. First, understand the ***base function*** $f(x)=x^2$ completely.
2. Then, learn how to ***transform*** the ***base function*** $f(x)=x^2$ into any other quadratic.

Complete the following table to ensure that you understand the base functions of a few common ***families of functions***.

<table>
<tr><th>Family of Functions</th><th>Base Function</th><th>Domain and Range</th><th>Table of Values</th><th>Graph</th><th>Intercepts</th></tr>
<tr><td>Linear</td><td>$f(x)=x$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–3</td><td></td></tr><tr><td>–2</td><td></td></tr><tr><td>–1</td><td></td></tr><tr><td>$-\frac{1}{2}$</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>$\frac{1}{2}$</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>3</td><td></td></tr></table></td><td></td><td></td></tr>
<tr><td>Quadratic</td><td>$f(x)=x^2$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–3</td><td></td></tr><tr><td>–2</td><td></td></tr><tr><td>–1</td><td></td></tr><tr><td>$-\frac{1}{2}$</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>$\frac{1}{2}$</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>3</td><td></td></tr></table></td><td></td><td></td></tr>
<tr><td>Cubic</td><td>$f(x)=x^3$</td><td></td><td><table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–3</td><td></td></tr><tr><td>–2</td><td></td></tr><tr><td>–1</td><td></td></tr><tr><td>$-\frac{1}{2}$</td><td></td></tr><tr><td>0</td><td></td></tr><tr><td>$\frac{1}{2}$</td><td></td></tr><tr><td>1</td><td></td></tr><tr><td>2</td><td></td></tr><tr><td>3</td><td></td></tr></table></td><td></td><td></td></tr>
</table>

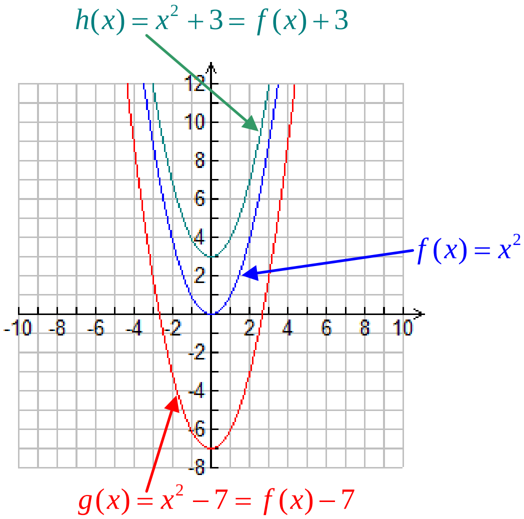
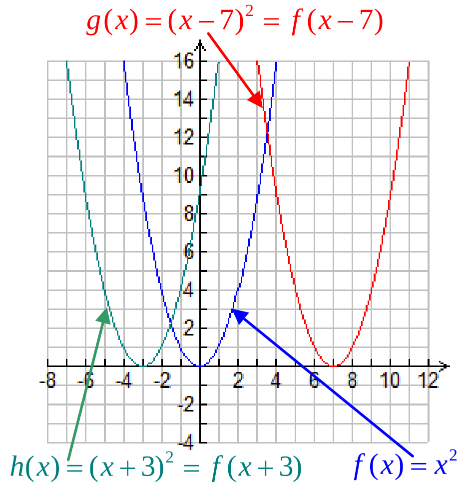
Family of Functions	*Base Function*	*Domain and Range*	*Table of Values*	*Graph*	*Intercepts*
Square Root	$f(x)=\sqrt{x}$		x | $f(x)$ -2 -1 $-\frac{1}{2}$ 0 1 4 6 9 12 14 16		
Reciprocal	$f(x)=\frac{1}{x}$		x | $f(x)$ -6 -4 -2 -1 $-\frac{1}{2}$ $-\frac{1}{4}$ 0 $\frac{1}{4}$ $\frac{1}{2}$ 1 2 4 6		
Absolute Value	$f(x)=\lvert x\rvert$		x | $f(x)$ -7 -4 -3 -2 -1 0 1 2 3 4 7		

Homework
pp. 182 – 183, #3, 7, 8, 9

TRANSFORMATION #1 – TRANSLATIONS OF FUNCTIONS

What is a Translation?

- A ***vertical translation*** of a function is obtained by ***adding a constant value to each value of the dependent variable*** of the given function. This results in the graph of the function "sliding" up or down, depending on whether the constant is positive or negative.
- A ***horizontal translation*** of a function is obtained by ***adding a constant value to each value of the independent variable*** of the given function. This results in the graph of the function "sliding" left or right, depending on whether the constant is positive or negative.

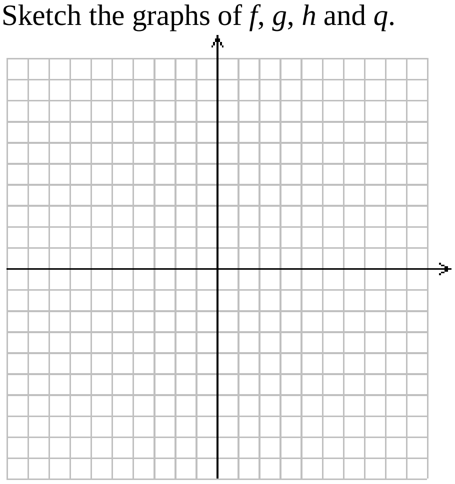
Example Vertical Translations	*Example Horizontal Translations*
	

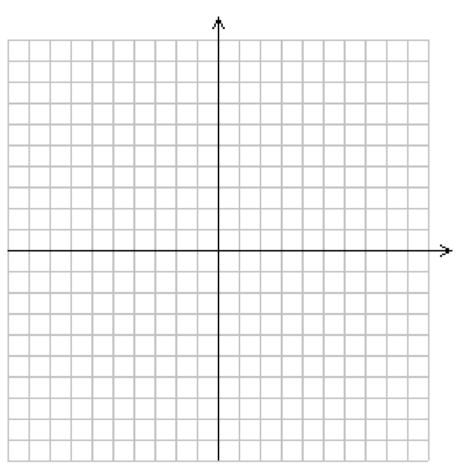
Understanding Translations of Functions

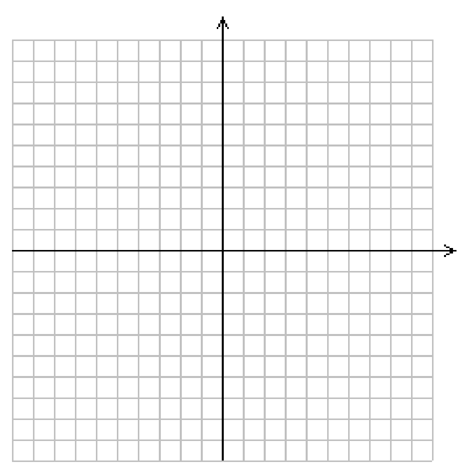
Complete the following table of values.

x	$f(x)=x^2$	$g(x)=x^2+3=f(x)+3$	$h(x)=(x+5)^2=f(x+5)$	$q(x)=(x+5)^2+3=f(x+5)+3$
–4				
–3				
–2				
–1				
0				
1				
2				
3				
4				

Sketch the graphs of *f*, *g*, *h* and *q*.



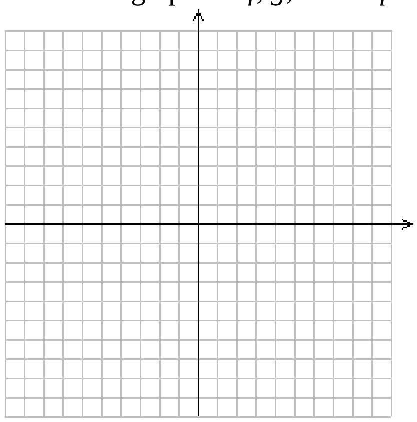
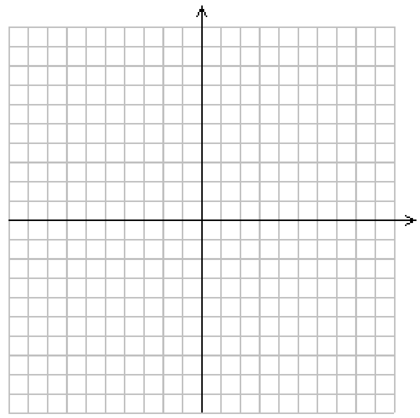
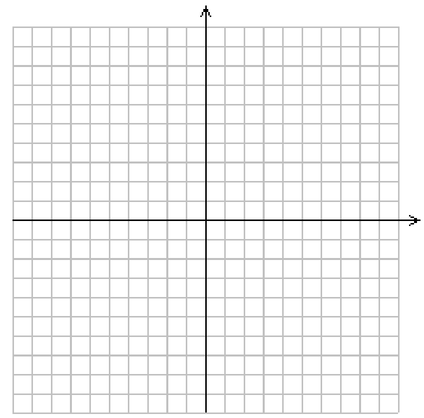




Complete the following table of values.

x	$f(x)=\frac{1}{x}$	$g(x)=\frac{1}{x}-2=f(x)-2$	$h(x)=\frac{1}{x-3}=f(x-3)$	$q(x)=\frac{1}{x-3}-2=f(x-3)-2$
-2				
-1				
$-\frac{1}{2}$				
$-\frac{1}{4}$				
0				
$\frac{1}{4}$				
$\frac{1}{2}$				
1				
2				

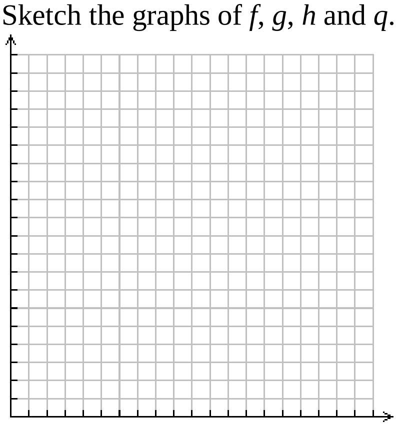
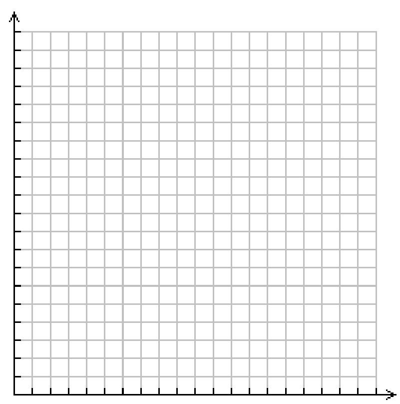
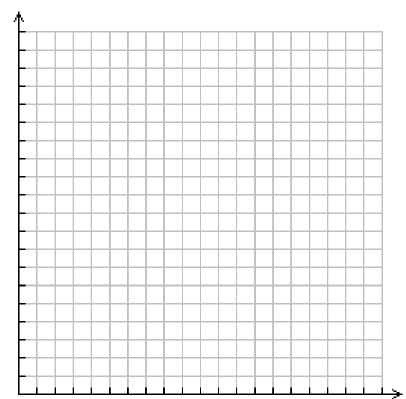
Sketch the graphs of *f*, *g*, *h* and *q*.

Complete the following table of values.

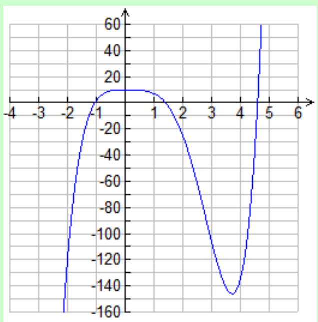
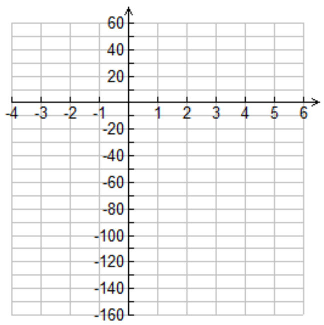
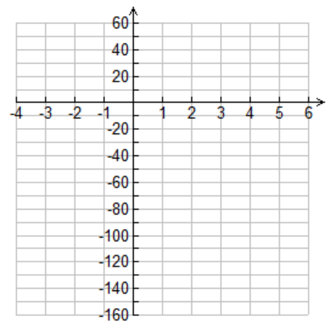
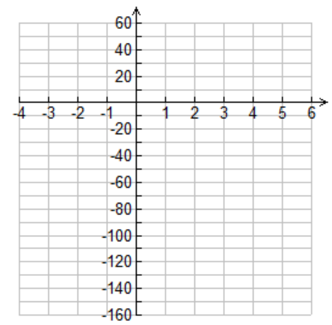
x	$f(x)=\sqrt{x}$	$g(x)=\sqrt{x}+1=f(x)+1$	$h(x)=\sqrt{x-4}=f(x-4)$	$q(x)=\sqrt{x-4}+1=f(x-4)+1$
-4				
-3				
-2				
-1				
0				
1				
2				
3				
4				

Sketch the graphs of *f*, *g*, *h* and *q*.

Analysis and Conclusions

Now carefully study the graphs and the tables on the two previous pages. Then complete the following table.

Base Function $y=f(x)$	***Translations of Base Function***		
	$y=f(x-h),\ h\in\mathbb{R}$	$y=f(x)+k,\ k\in\mathbb{R}$	$y=f(x-h)+k,\ h\in\mathbb{R},\ k\in\mathbb{R}$
Description of Translations			
$y=f(x)$	$y=f(x+2)$	$y=f(x)+20$	$y=f(x-1)-10$
			

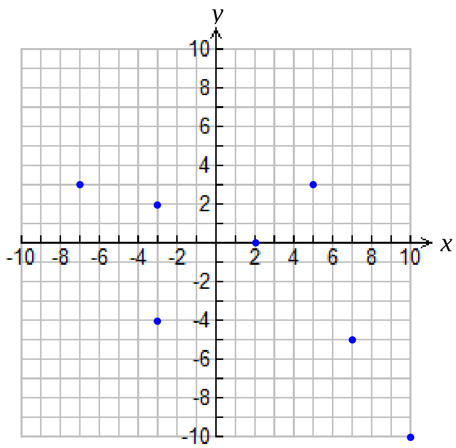
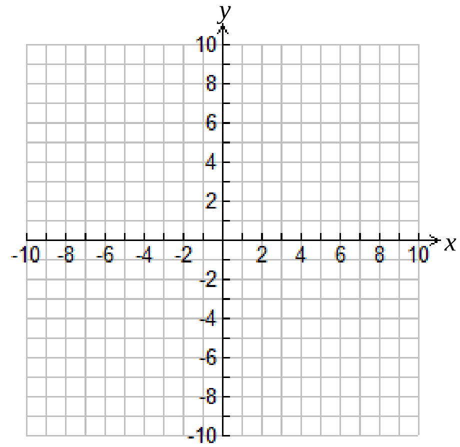
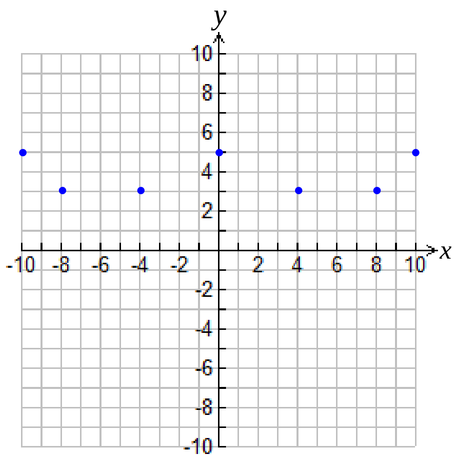
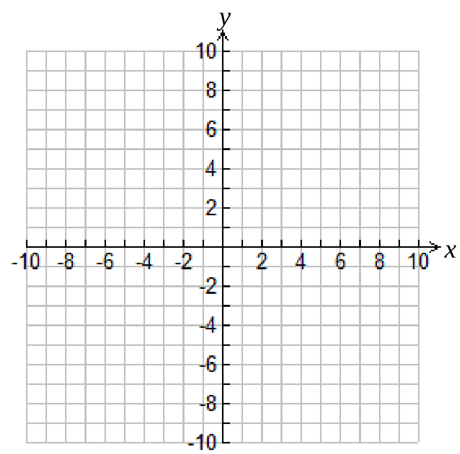
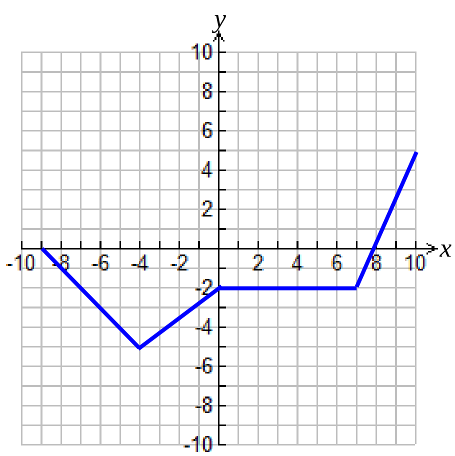
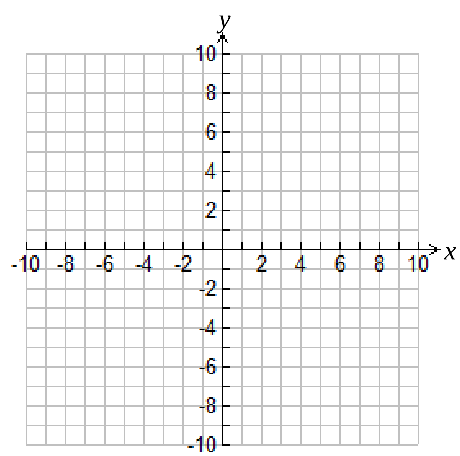
Extremely Important Questions and Exercises…

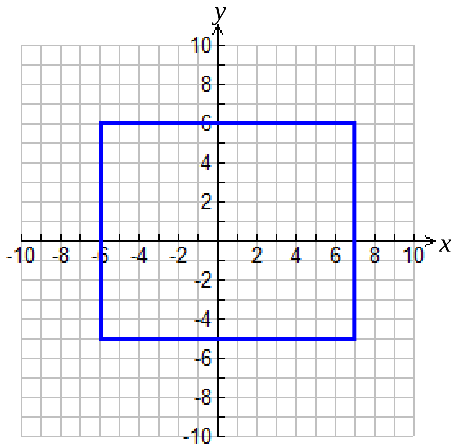
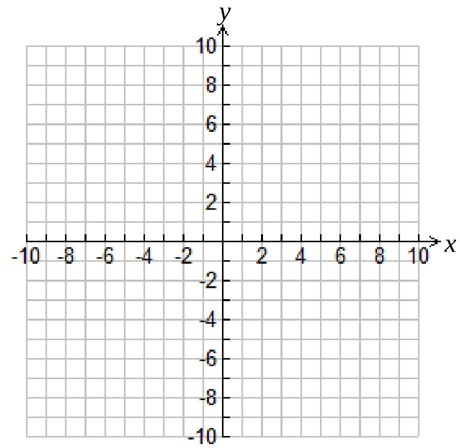
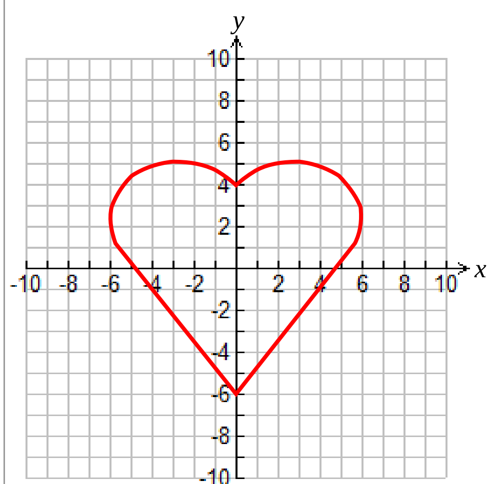
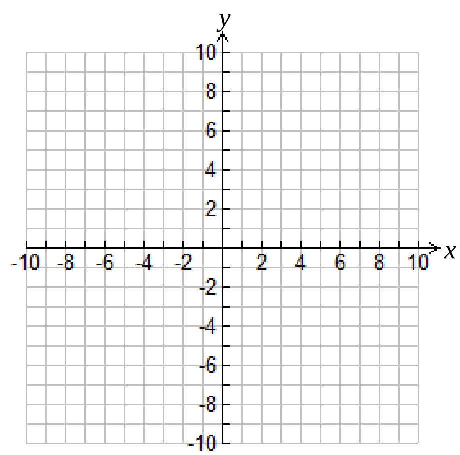
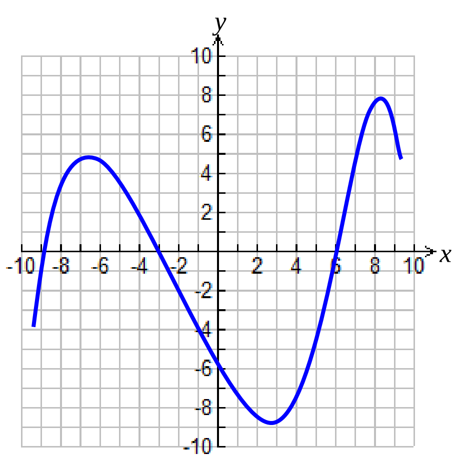
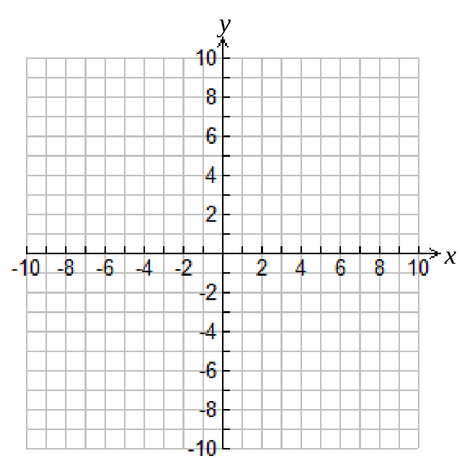
1. So far we have considered both ***horizontal*** and ***vertical*** translations of functions. Does it matter in which order these translations are performed?

2. Complete the following table.

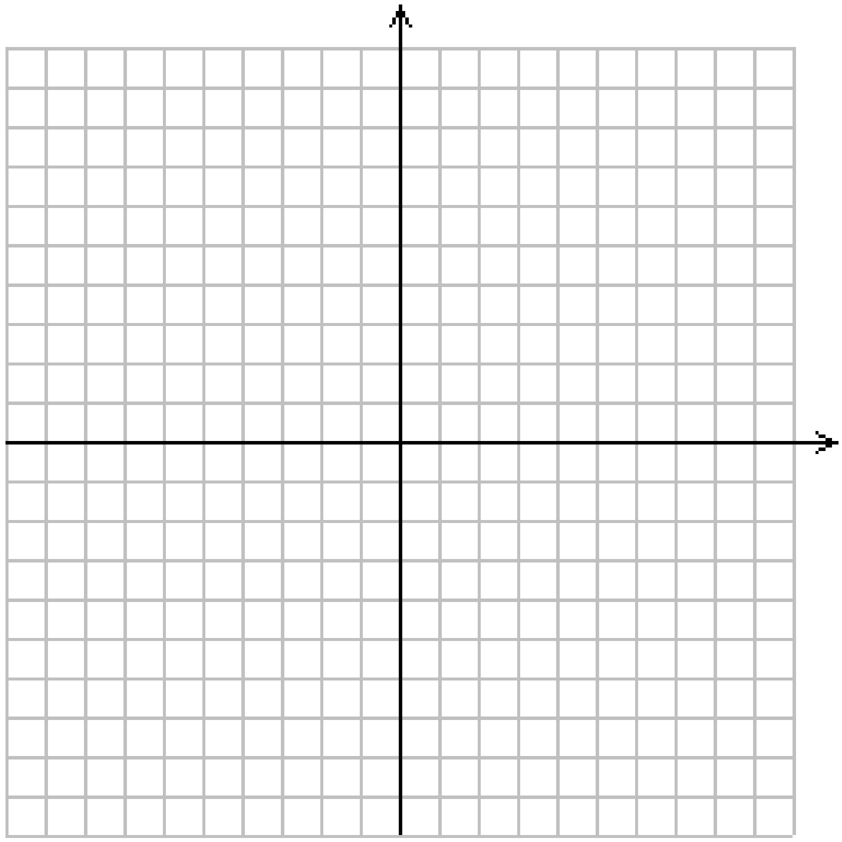
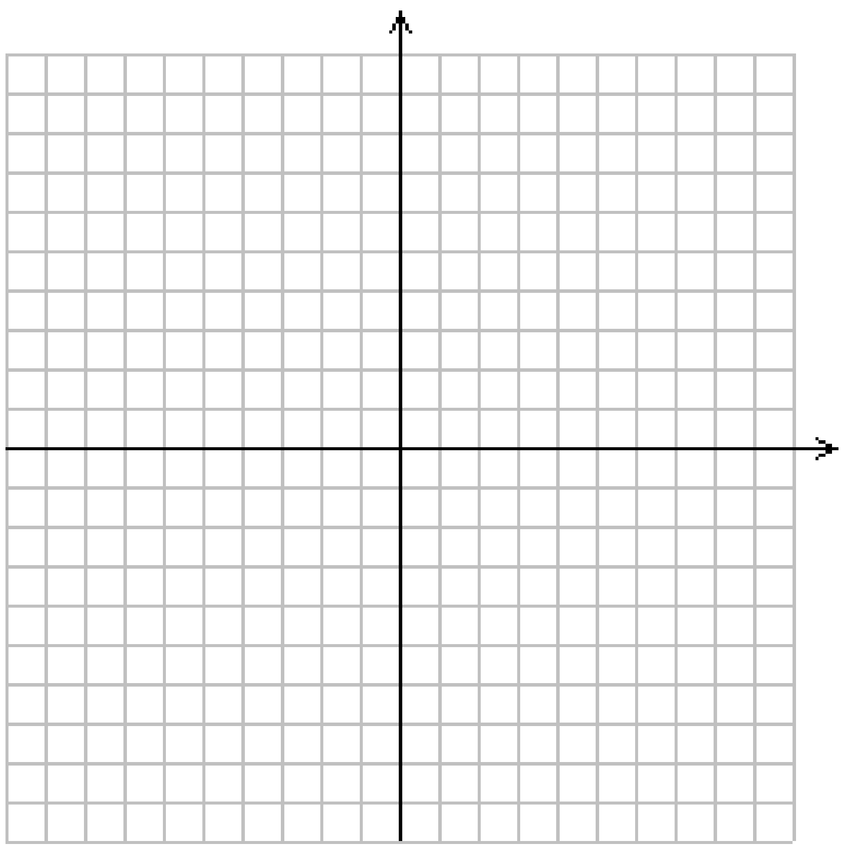
Function	***Base Function***	***Translation(s) of Base Function Required to obtain Function***
(a) $f(x)=x+16$		There are two correct answers for this one.
(b) $g(x)=x^2-5x+6$		
(c) $h(x)=\sqrt{x-6}+5$		
(d) $p(x)=\dfrac{1}{x+2}-5$		
(e) $q(x)=x^3+3x^2+3x+1$		

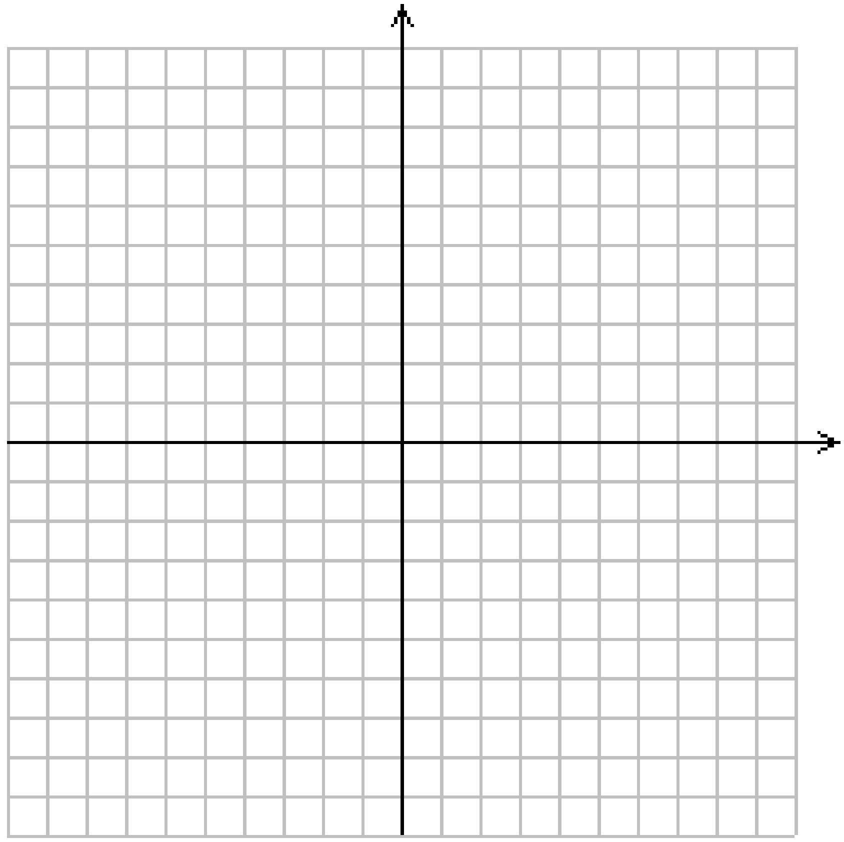
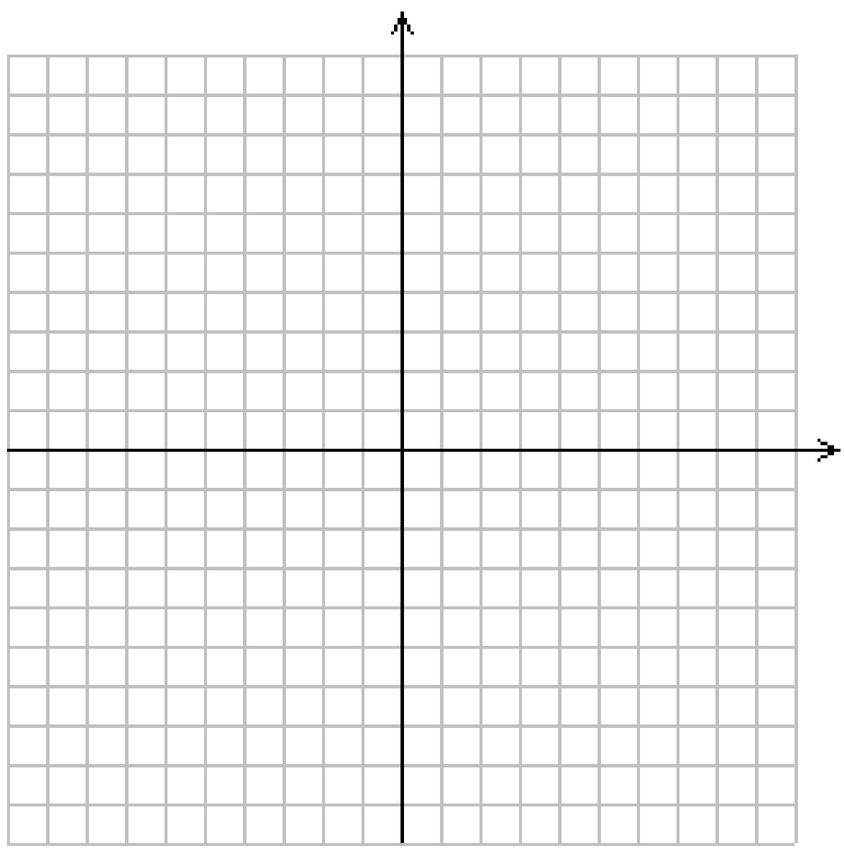
3. Complete the following table.

Graph of Relation	*Relation or Function?*	*Discrete or Continuous?*	*Graph of Translated Relation*
			Translate 3 units up and 2 to the left.  If the given relation is a function $y = f(x)$, then this is ______________
			Translate 10 units down.  If the given relation is a function $y = f(x)$, then this is ______________
			Translate 2 units down and 1 to the left.  If the given relation is a function $y = f(x)$, then this is ______________

Graph of Relation	*Relation or Function?*	*Discrete or Continuous?*	*Graph of Translated Relation*
			Translate 3 units down and 2 to the right.  If the given relation is a function $y = f(x)$ then this is _______________
			Translate 1 unit down and 4 to the right.  If the given relation is a function $y = f(x)$ then this is _______________
			Translate 2 units up and 1 to the right.  If the given relation is a function $y = f(x)$ then this is _______________

4. Without using a table of values, graph the following functions on the same grid. In addition, state the domain and range of each function.

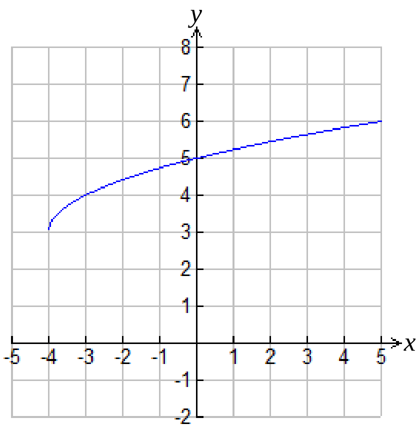
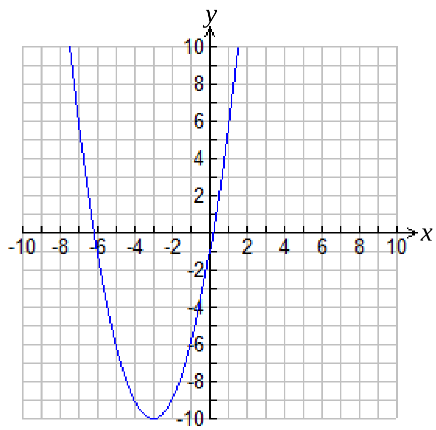
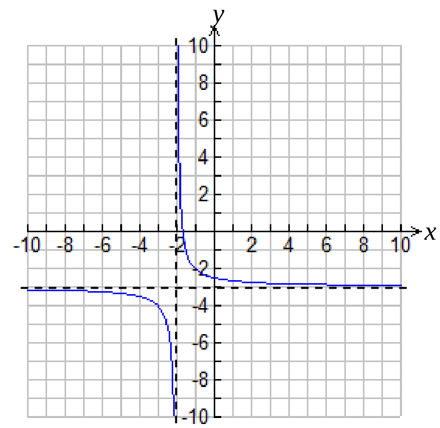
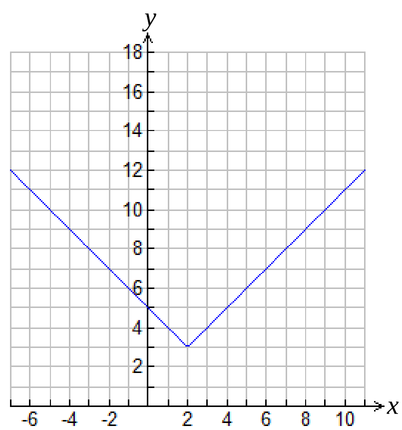
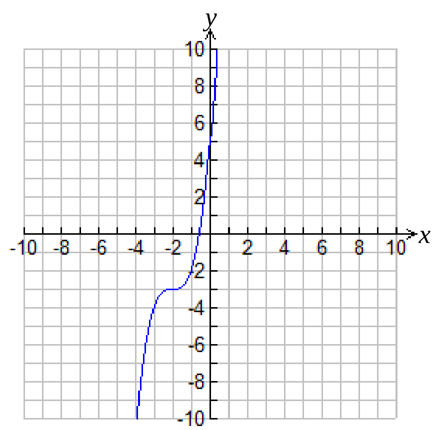
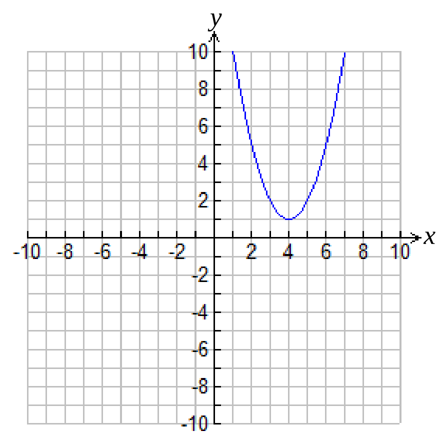
Functions	***Graphs***	***Domain and Rage***
$y=\frac{1}{x}+2$ $y=\frac{1}{x+2}$ $y=\frac{1}{x-3}-4$		
$y=\sqrt{x}+5$ $y=\sqrt{x+5}$ $y=\sqrt{x+7}-4$		

Functions	*Graphs*	*Domain and Range*
$y = x^2 + 3$ $y = (x+3)^2$ $y = (x+5)^2 - 10$		
$y = \|x\| - 6$ $y = \|x - 6\|$ $y = \|x+5\| - 7$		

5. Without graphing, state the domain, range, intercepts (if any) and asymptotes (if any) of each given function.

Function	***Domain and Range***	***Intercepts***	***Asymptotes***
$f(x)=x^2+5$			
$g(x)=\sqrt{x}-10$			
$h(x)=x^3-15$			
$s(t)=\dfrac{1}{t+3}-15$			
$p(y)=\lvert y-8\rvert+15$			

6. For each graph, state an equation that best describes it.

Homework

pp. 189 – 192, 1 $\rightarrow$ 5, 6f, 7e,f, 8e,f, 10, 12, 13,14, 16, 18

Transformation #2 – Reflections of Functions

What is a Reflection?

A ***reflection of a relation in a given line*** is produced by replacing ***each point*** in the given relation by a point ***symmetrically placed*** on the other side of the line. Intuitively, a reflection of a point in a line is the ***mirror image*** of the point about the line. Think of what you see when you look at yourself in a mirror. First of all, your ***reflection*** appears to be ***behind*** the surface of the mirror. Moreover, its ***distance*** from the mirror appears to be exactly the same as your distance from the mirror.

Examples of Reflections

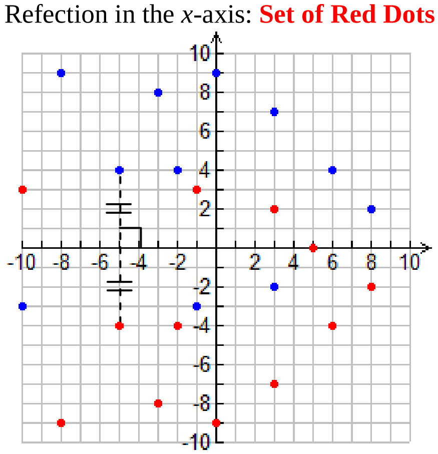
Relation

x	y
–10	–2
–8	9
–5	4
–3	8
–2	4
–1	–3
0	9
3	7
3	–2
6	4
8	2

Reflection in x-axis

x	y
–10	2
–8	–9
–5	–4
–3	–8
–2	–4
–1	3
0	–9
3	–7
3	2
6	–4
8	–2

Original Relation: **Set of Blue Dots**
Refection in the x-axis: **Set of Red Dots**



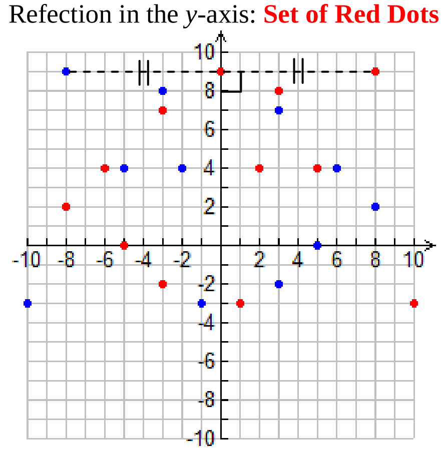
Relation

x	y
–10	–2
–8	9
–5	4
–3	8
–2	4
–1	–3
0	9
3	7
3	–2
6	4
8	2

Reflection in y-axis

x	y
10	–2
8	9
5	4
3	8
2	4
1	–3
–0	9
–3	7
–3	–2
–6	4
–8	2

Original Relation: **Set of Blue Dots**
Refection in the y-axis: **Set of Red Dots**



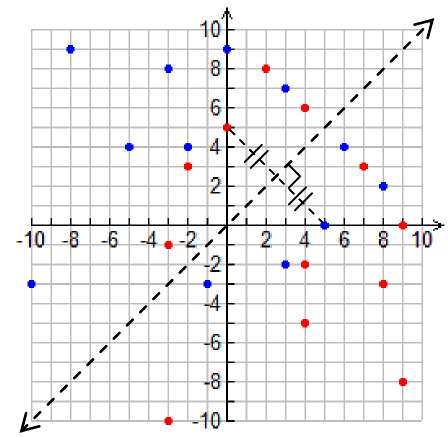
Relation

x	y
–10	–2
–8	9
–5	4
–3	8
–2	4
–1	–3
0	9
3	7
3	–2
6	4
8	2

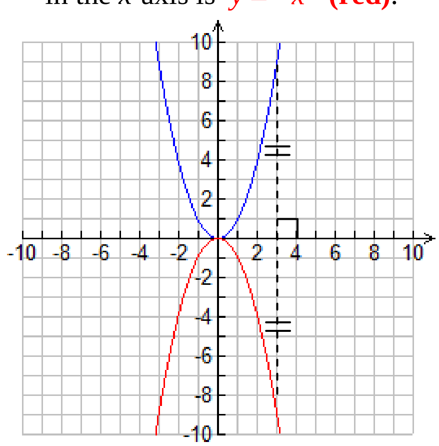
Reflection in y=x

x	y
–2	–10
9	–8
4	–5
8	–3
4	–2
–3	–1
9	0
7	3
–2	3
4	6
2	8

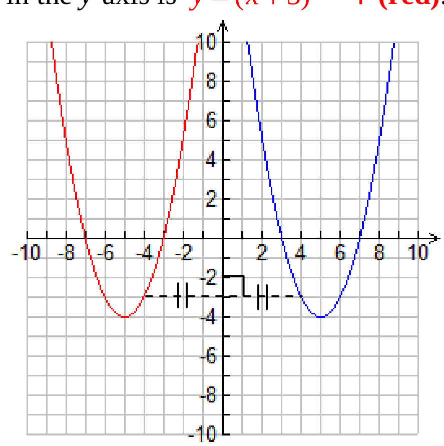
Original Relation: **Set of Blue Dots**
Refection in line $y = x$: **Set of Red Dots**



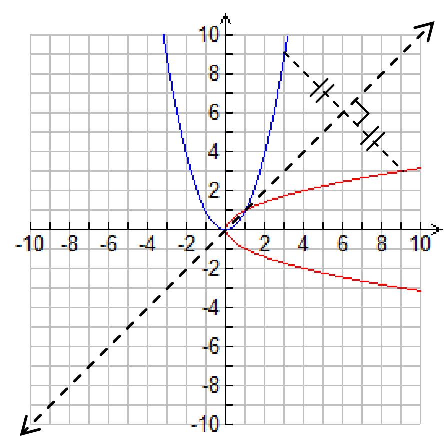
The reflection of $y = x^2$ **(blue)** in the x-axis is $y = -x^2$ **(red)**.



The reflection of $y = (x-5)^2 - 4$ **(blue)** in the y-axis is $y = (x+5)^2 - 4$ **(red)**.



The reflection of $y = x^2$ **(blue)** in the line $y = x$ is $y = \pm\sqrt{x}$ **(red)**.



Important Questions

1. Suppose that the point Q is the reflection of the point P in the line l. Suppose further that the line segment PQ intersects the line l at the point A. What can you conclude about the lengths of the line segments PA and QA? Draw a diagram to illustrate your answer.

2. Is the reflection of a function in the x-axis also a function? Explain.

3. Is the reflection of a function in the y-axis also a function? Explain.

4. Is the reflection of a function in the line $y = x$ also a function? Explain.

5. Suppose that R represents a relation that is ***not*** a function. Can a reflection of R be a function? If so, give examples.

Investigation

You may use a graphing calculator or graphing software such as TI-Interactive to complete the following table.

Function	***Equation of*** $-f(x)$	***Equation of*** $f(-x)$	***Equation of*** $-f(-x)$	***Graph of*** $y=-f(x)$	***Graph of*** $y=f(-x)$	***Graph of*** $y=-f(-x)$
$f(x)=x$						
$f(x)=x^2$						
$f(x)=x^3$						

Continued on next page…

Function	***Equation of*** $-f(x)$	***Equation of*** $f(-x)$	***Equation of*** $-f(-x)$	***Graph of*** $y=-f(x)$	***Graph of*** $y=f(-x)$	***Graph of*** $y=-f(-x)$
$f(x)=\sqrt{x}$						
$f(x)=\dfrac{1}{x}$						
$f(x)=\lvert x\rvert$						

Conclusions

Given a function *f*,

- the graph of $y=-f(x)$ is the ______________________ of the graph of $y=f(x)$ in the ______________________.
- the graph of $y=f(-x)$ is the ______________________ of the graph of $y=f(x)$ in the ______________________.
- the graph of $y=-f(-x)$ is the ______________________ of the graph of $y=f(x)$ in the ______________________ followed by the ______________________ of the graph of ________________ in the ______________________.

Important Terminology

- Suppose that a transformation is applied to a point *P* to obtain the point *Q*. Then, *P* is called the ***pre-image of Q*** and *Q* is called the ***image of P***.
- If *Q* is the image of *P* under some transformation and *P* = *Q*, then *P* is said to be ***invariant*** under the transformation.

Homework

Read pp. 196 – 202

pp. 203 – 207, #1, 2, 3, 4cd, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 22, 23

Combinations of Translations and Reflections

Overview

Base Function: $y = f(x)$ (e.g. $f(x)=\sqrt{x}$)			
Translations of Base Function		***Reflections of Base Function***	
Horizontal	***Vertical***	***In y-axis (Horizontal)***	***In x-axis (Vertical)***
$y=g(x)=f(x-h)$ $\therefore (x,y)\to(x+h,y)$	$y=g(x)=f(x)+k$ $\therefore (x,y)\to(x,y+k)$	$y=g(x)=f(-x)$ $\therefore (x,y)\to(-x,y)$	$y=g(x)=-f(x)$ $\therefore (x,y)\to(x,-y)$
e.g. $g(x)=\sqrt{x+3}$ $\therefore (x,y)\to(x-3,y)$	**e.g.** $g(x)=\sqrt{x}-2$ $\therefore (x,y)\to(x,y-2)$	**e.g.** $g(x)=\sqrt{-x}$ $\therefore (x,y)\to(-x,y)$	**e.g.** $g(x)=-\sqrt{x}$ $\therefore (x,y)\to(x,-y)$
Combinations			
Translations		***Reflections***	
Horizontal combined with Vertical		***In x-axis (vertical) combined with y-axis (horizontal)***	
$y=g(x)=f(x-h)+k$	$\therefore (x,y)\to(x+h,y+k)$	$y=g(x)=-f(-x)$	$\therefore (x,y)\to(-x,-y)$
e.g. $g(x)=\sqrt{x+3}-2$	$\therefore (x,y)\to(x-3,y-2)$	**e.g.** $g(x)=-\sqrt{-x}$	$\therefore (x,y)\to(-x,-y)$
Translations and Reflections			
$y=g(x)=-f(-(x-h))+k$	$\therefore (x,y)\to(-x+h,-y+k)$		
e.g. $g(x)=-f(-(x+3))-2=-\sqrt{-(x+3)}-2$	$\therefore (x,y)\to(-x-3,-y-2)$		

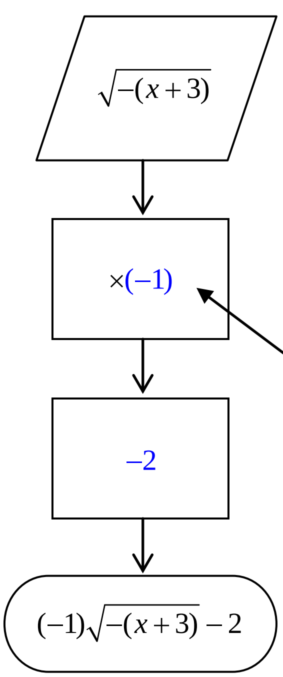
To understand this combination of translations and reflections, it is necessary to

1. treat the vertical and horizontal transformations ***separately***.
2. understand the ***order of operations***.

$$\textbf{e.g. } g(x)=-\sqrt{-(x+3)}-2=(-1)\sqrt{(-1)(x+3)}-2$$

Vertical

$$(-1)\sqrt{-(x+3)}-2$$

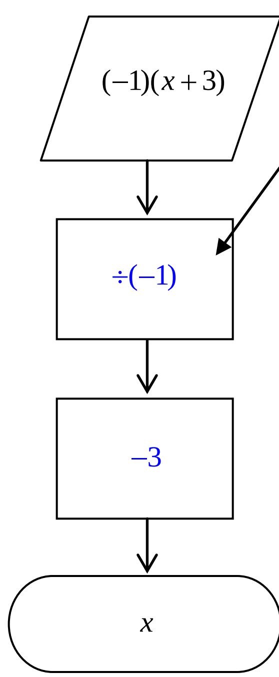


Because of the order of operations, $\sqrt{-(x+3)}$ must be multiplied by -1 ***before*** 2 is subtracted.

Therefore, the reflection in the x-axis must be performed ***before*** the vertical translation down 2 units.

Horizontal

$$\sqrt{(-1)(x+3)}$$



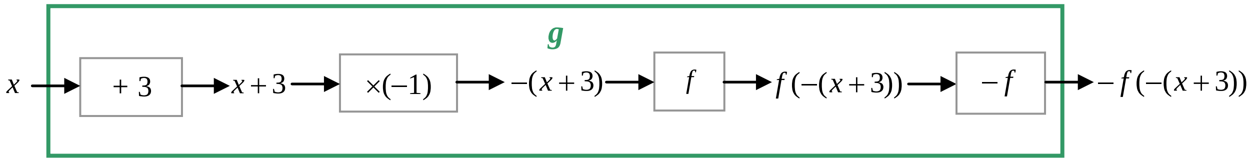
Keep in mind that $-(x+3)$ is the "x-value" used as the input for the function f. However, for the function g defined by $g(x)=-f(-(x+3))-2$, the input is still x. This means that given the value of $-(x+3)$, we need to ***reverse the operations to find x*** (in the order opposite the order of operations).

Therefore, the transformations must be performed in the order ***opposite*** the order of operations. ***First*** f is ***reflected in the y-axis*** and ***then*** the resulting function is translated ***3 units left***.

Note that division by -1 is the same as multiplication by -1, which is why the transformation is still a reflection.

Deepening your Understanding – A Machine View of $g(x) = -f(-(x+3))$

If you found the discussion on the previous page a little daunting, it may be helpful to return to the machine analogy. We can think of the function *g* as a single machine that consists of several simpler machines. If the input "*x*" is given to *g*, through a series of small steps, the output $-f(-(x+3))$ is eventually produced.



This diagram should help you to understand why everything works backwards for horizontal transformations. Although we must evaluate *f* at $-(x+3)$ to obtain the value of *g*(*x*), *g* is still evaluated at *x*. Therefore, we must work our way in reverse from $-(x+3)$ to *x* to understand the horizontal transformations applied to *f* to produce *g*.

Summary

To obtain the graph of $y = g(x) = -f(-(x-h)) + k = (-1)f((-1)(x-h)) + k$ from the graph of $y = f(x)$, perform the following transformations.

Vertical (Follow the order of operations)

1. First ***reflect*** in the *x*-axis.
2. Then ***translate*** *k* units ***up*** if $k > 0$ or *k* units ***down*** if $k < 0$.

Horizontal (Reverse the operations in the order opposite the order of operations)

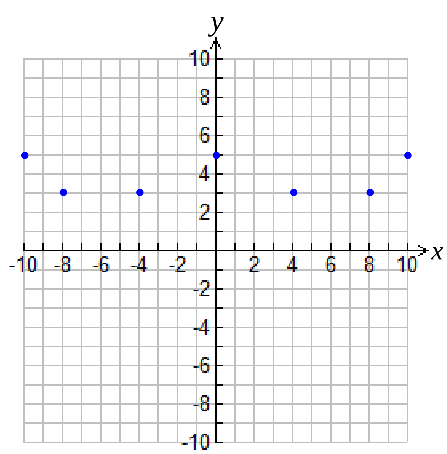
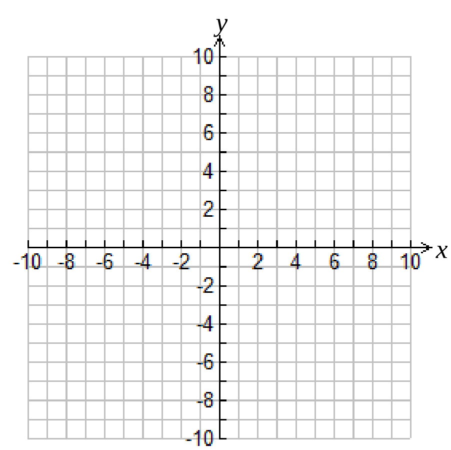
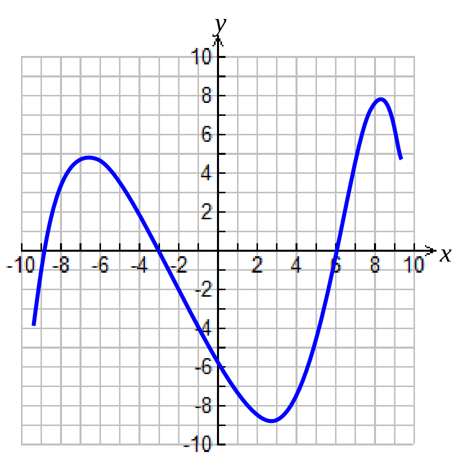
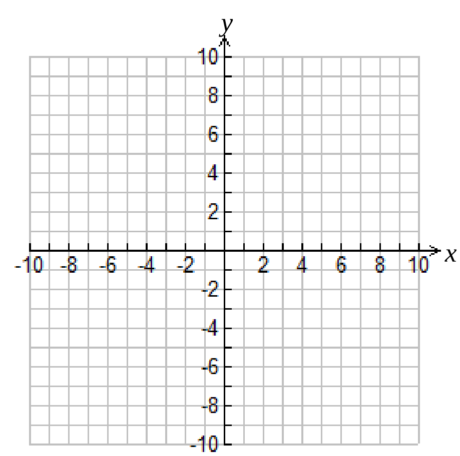
1. First ***reflect*** in the *y*-axis.
2. Then ***translate*** *h* units ***right*** if $h > 0$ or *h* units ***left*** if $h < 0$.

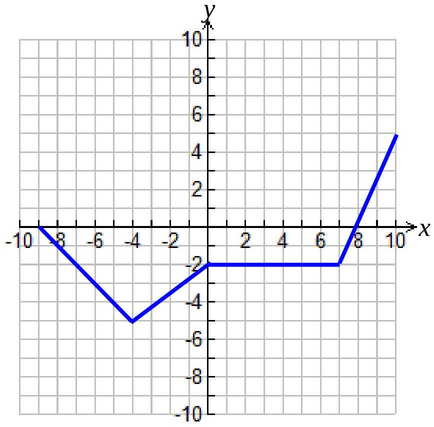
Summary

$\therefore (x, y) \rightarrow (-x+h, -y+k)$

Important Exercises

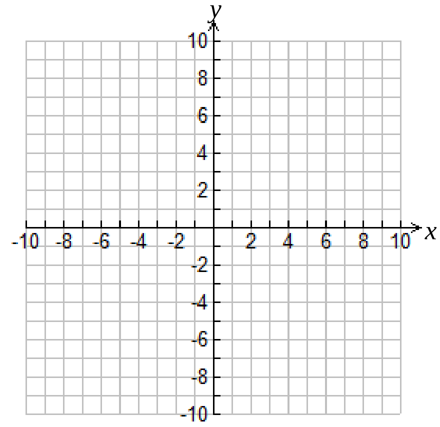
Perform the specified transformations on the graph of each given function.

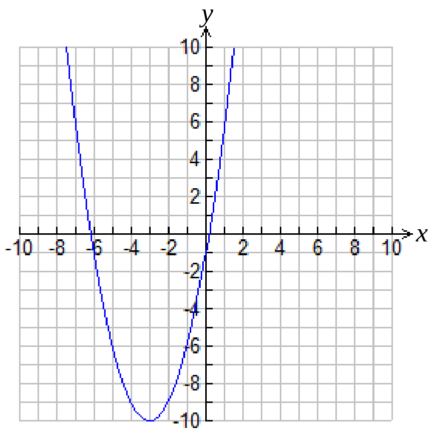
Graph of Function $y = f(x)$	*Transformation of f*	*Graph of Transformed Function* $y = g(x)$
	$g(x) = -f(x) - 2$ Explain this transformation of *f* in words.	
	$g(x) = -f(x-1) + 2$ Explain this transformation of *f* in words.	



$g(x)= f(-(x+3))-4$

Explain this transformation of *f* in words.





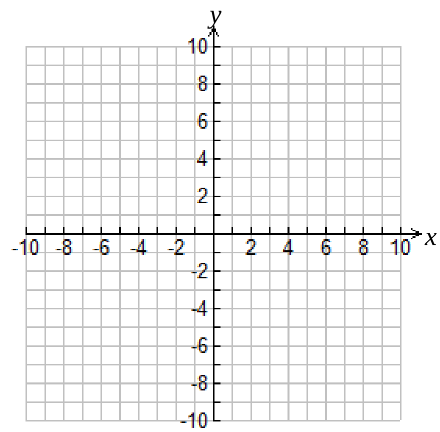
$g(x)=-f(-(x-3))+6$

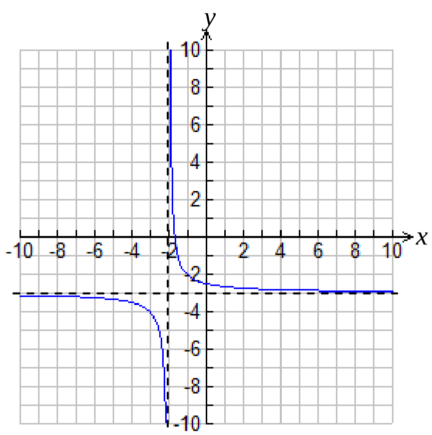
Explain this transformation of *f* in words.

Possible equations of *f* and *g*.

f(*x*) =

g(*x*) =





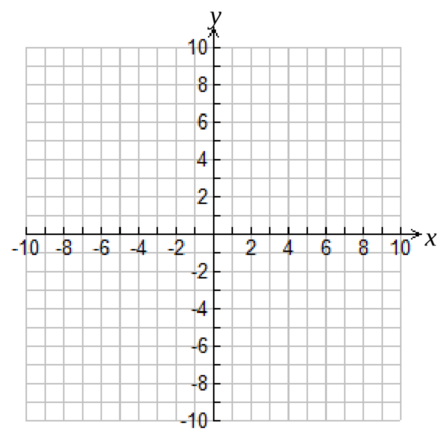
$g(x)=-f(-(x-5))+8$

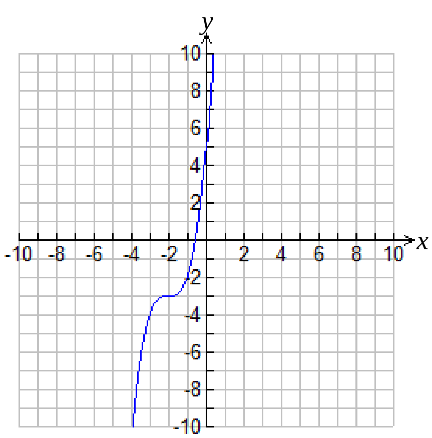
Explain this transformation of *f* in words.

Possible equations of *f* and *g*.

f(*x*) =

g(*x*) =





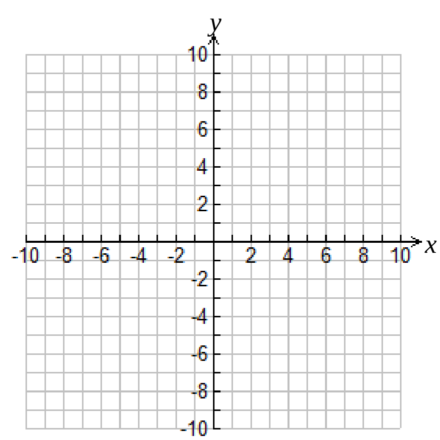
$g(x)= f(-x-5)+3$

Explain this transformation of *f* in words. ***(Be careful with this one.)***

Possible equations of *f* and *g*.

f(*x*) =

g(*x*) =



Transformation #3: Stretches and Compressions of Functions

Terminology

Dilatation, Dilation, Expansion or Stretch of a Function

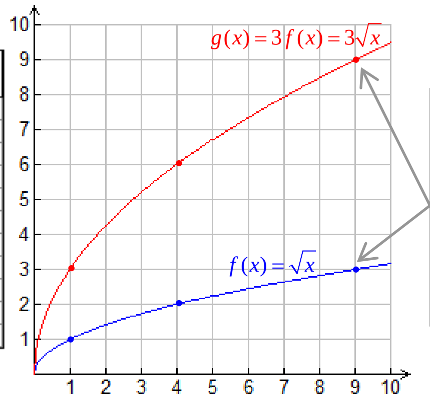
A transformation in which all distances on the co-ordinate plane are ***lengthened*** by multiplying either all x-co-ordinates (horizontal dilation) or all y-co-ordinates (vertical dilation) by a common factor greater than 1.

Compression of a Function

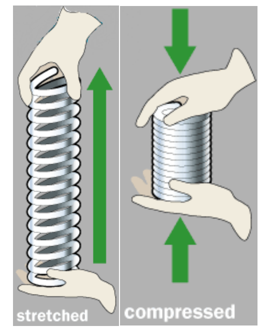
A transformation in which all distances on the co-ordinate plane are ***shortened*** by multiplying either all x-co-ordinates (horizontal compression) or all y-co-ordinates (vertical compression) of a graph by a common factor less than 1.

Examples

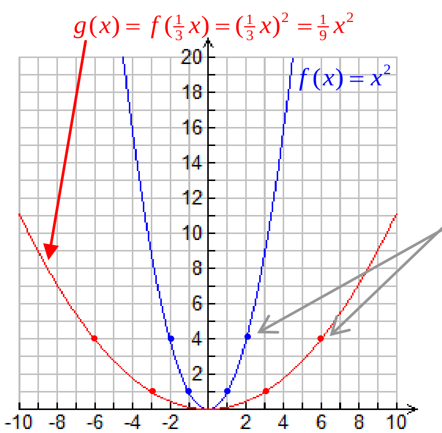
x	y1(x) sqrt(x)	y2(x) 3sqrt(x)
0	0	0
1	1	3
2	1.41421	4.24264
3	1.73205	5.19615
4	2	6
5	2.23607	6.7082
6	2.44949	7.34847
7	2.64575	7.93725
8	2.82843	8.48528
9	3	9
10	3.16228	9.48683



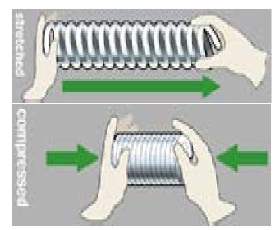
Notice that the y-co-ordinates of points lying on the graph of the function g are three times the corresponding y-co-ordinates of points lying on f. We say that f is ***stretched vertically by a factor of 3*** to produce g.



x	y1(x) x^2	y2(x) (1/3x)^2
0	0	0
1	1	0.111111
2	4	0.444444
3	9	1
4	16	1.77778
5	25	2.77778
6	36	4
7	49	5.44444
8	64	7.11111
9	81	9
10	100	11.1111

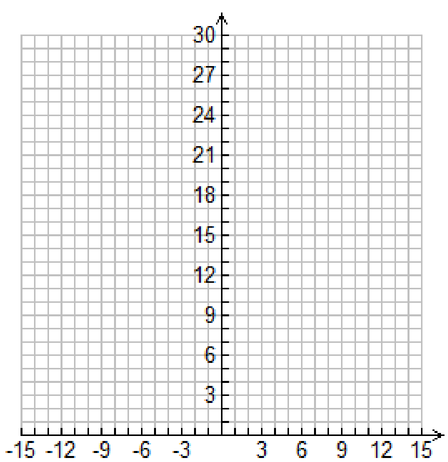


Notice that the x-co-ordinates of points lying on the graph of the function g are three times the x-co-ordinates of corresponding points lying on f. We say that f is ***stretched horizontally by a factor of 3*** to produce g.

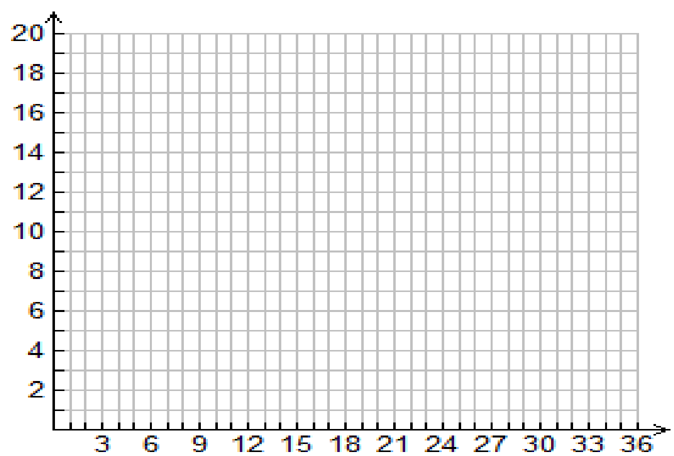


Investigation

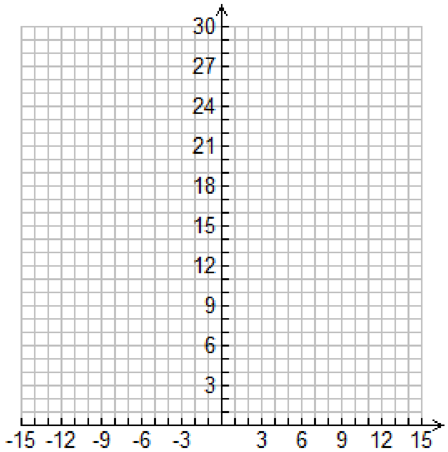
1. Sketch $f(x)=x^2$, $g(x)=f(2x)=(2x)^2=4x^2$ and $h(x)=f(\frac{1}{2}x)=(\frac{1}{2}x)^2=\frac{1}{4}x^2$ on the same grid. Describe how the graphs of g and h are related to the graph of f.



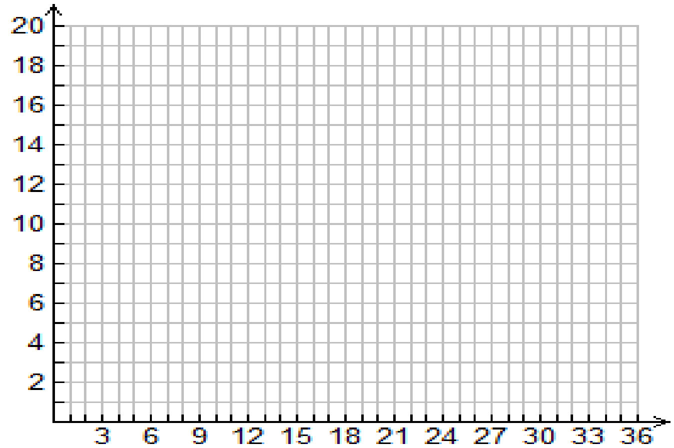
2. Sketch $f(x)=\sqrt{x}$, $g(x)=f(2x)=\sqrt{2x}$ and $h(x)=f(\frac{1}{2}x)=\sqrt{\frac{1}{2}x}$ on the same grid. Describe how the graphs of g and h are related to the graph of f.



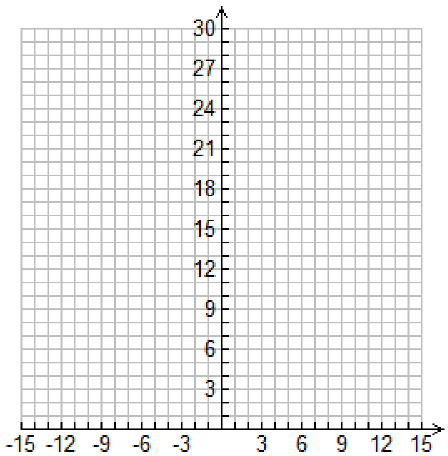
3. Sketch $f(x)=x^2$, $g(x)=2f(x)=2x^2$ and $h(x)=\frac{1}{2}f(x)=\frac{1}{2}x^2$ on the same grid. Describe how the graphs of *g* and *h* are related to the graph of *f*.



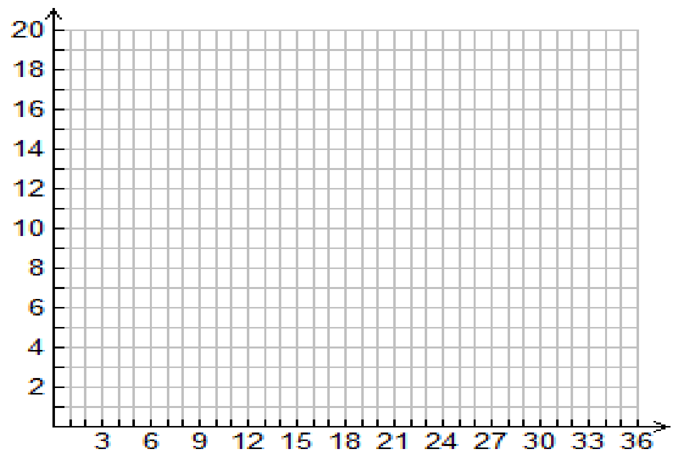
4. Sketch $f(x)=\sqrt{x}$, $g(x)=2f(x)=2\sqrt{x}$ and $h(x)=\frac{1}{2}f(x)=\frac{1}{2}\sqrt{x}$ on the same grid. Describe how the graphs of *g* and *h* are related to the graph of *f*.



5. Sketch $f(x)=x^2$ and $g(x)=2f(3x)=$ ______________. Describe how the graph of *g* is related to the graph of *f*.



6. Sketch $f(x)=\sqrt{x}$ and $g(x)=3f(\frac{1}{2}x)=$ ______________. Describe how the graph of *g* is related to the graph of *f*.



Summary

- To obtain the graph of $y=g(x)=af(x), a\in\mathbb{R}$ from the graph of $y=f(x)$ ____.
- To obtain the graph of $y=g(x)=f(bx), b\in\mathbb{R}$ from the graph of $y=f(x)$ ____.
- To obtain the graph of $y=g(x)=af(bx), a\in\mathbb{R}, b\in\mathbb{R}$ from the graph of $y=f(x)$ ____.

Extremely Important Question

In the transformation $y=g(x)=af(bx), a\in\mathbb{R}, b\in\mathbb{R}$ of $y=f(x)$, what happens if $a=-1$ or $b=-1$? What can you conclude from this?

Homework

Read pp. 221 – 228
pp. 229 – 232, 1bdg, 2f, 3, 4iv, 6, 7, 8, 9, 10d, 11df, 12, 13, 14, 16

Putting it all Together – Combinations of Translations and Stretches

Summary

By now we have acquired enough knowledge to combine all the transformations.

To obtain the graph of $y = g(x) = af(b(x-h)) + k$ from the graph of $y = f(x)$, perform the following transformations.

Vertical (Follow the order of operations)

1. First ***stretch*** or ***compress vertically*** by the factor *a*. $(x,y) \to (x,ay)$

   If $|a| > 1$, ***stretch*** by a factor of *a*. ($|a| > 1$ is a short form for "$a > 1$ or $a < -1$")

   If $0 < |a| < 1$, ***compress*** by a factor of *a*.

   If $a < 0$ (i.e. *a* is negative), the stretch or compression is combined with a ***reflection***.
2. Then ***translate*** *k* units ***up*** if $k > 0$ or *k* units ***down*** if $k < 0$. $(x,y) \to (x,y+k)$

Horizontal (Reverse the operations in the order opposite the order of operations)

1. First ***stretch*** or ***compress horizontally*** by the factor $\frac{1}{b} = b^{-1}$. $(x,y) \to (b^{-1}x,y)$

   If $|b| > 1$, ***compress*** by the factor b^{-1}.

   If $0 < |b| < 1$, ***stretch*** by the factor b^{-1}.

   If $b < 0$ (i.e. *b* is negative), the stretch or compression is combined with a ***reflection***.
2. Then ***translate*** *h* units ***right*** if $h > 0$ or *h* units ***left*** if $h < 0$. $(x,y) \to (x+h,y)$

Summary using Mapping Notation

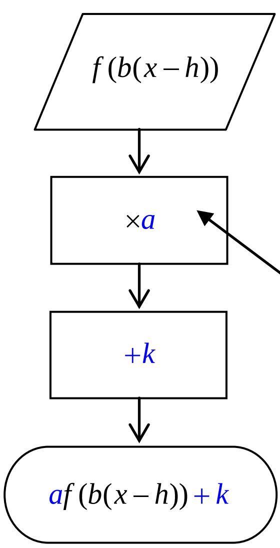
The following shows how an ordered pair belonging to *f* (pre-image) is mapped to an ordered pair belonging to *g* (image) under the transformation defined by $g(x) = af(b(x-h)) + k$.

pre-image → image $(x,y) \to (b^{-1}x+h, ay+k)$

Explanation

Vertical

$af(b(x-h)) + k$



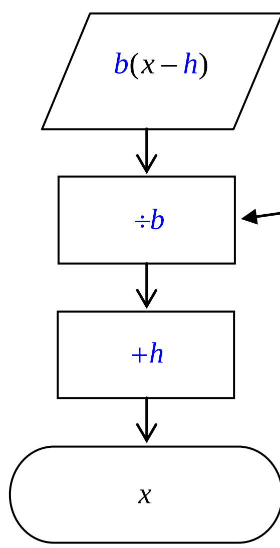
Because of the order of operations, $f(b(x-h))$ must be multiplied by *a* ***before*** *k* is added.

Therefore, the vertical stretch by a factor of *a* must be performed ***before*** the vertical translation by *k* units.

$(x,y) \to (x,ay+k)$

Horizontal

$b(x-h)$



To find the output of *g* when the input *x* is given, we look at the output of *f* when the input $b(x-h)$ is given.

Therefore, the transformations must be performed in the reverse order, ***opposite*** the order of operations. ***First f*** is ***stretched or compressed horizontally by a factor of b^{-1}*** and ***then*** the resulting function is translated ***h units right if h>0*** or ***h units left if h<0***.

$(x,y) \to (b^{-1}x+h,y)$

Example 1

Given $f(x)=(x-2)^2-3$, sketch the graph of $g(x)=-2f(3x-6)+3$.

Solution

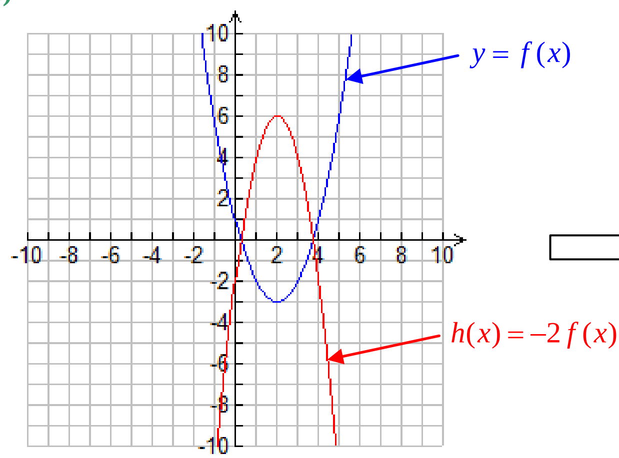
1. First of all, we should rewrite the transformation so that it matches the standard form $g(x)=af(b(x-h))+k$. This means that we need to factor $3x-6$. Then, $g(x)=-2f(3x-6)+3=-2f(3(x-2))+3$.

2. Next we list all the transformations, remembering to treat the vertical and horizontal transformations separately.

$a=-2$, $b=3$, $h=2$, $k=3$ $\quad\therefore (x,y)\to(\frac{1}{3}x+2,-2y+3)$	
Vertical (Follow order of operations)	***Horizontal (Reverse the order of operations)***
(i) Stretch vertically by a factor of $a=-2$. (Stretch by a factor of 2, then reflect in the x-axis.)	**(i)** Compress horizontally by a factor of $b^{-1}=1/3$.
(ii) Shift up by $k=3$ units.	**(ii)** Shift right by $h=2$ units.

3. Finally, we perform the transformations. (At each step, watch how the vertex is mapped from pre-image to image.)

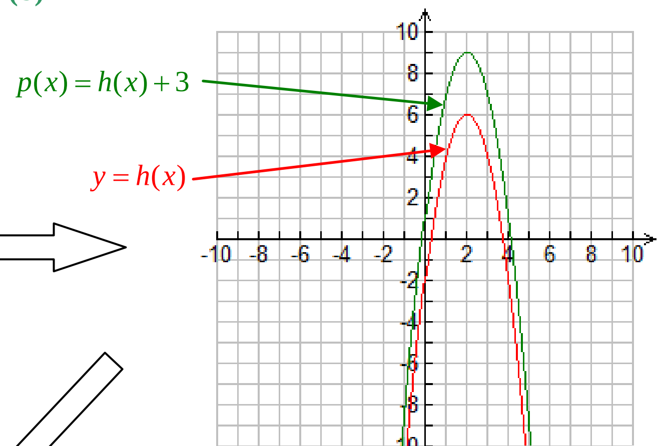
(a)



$(x,y)\to(x,-2y)$

e.g. $(2,-3)\to(2,-2(-3))=(2,6)$

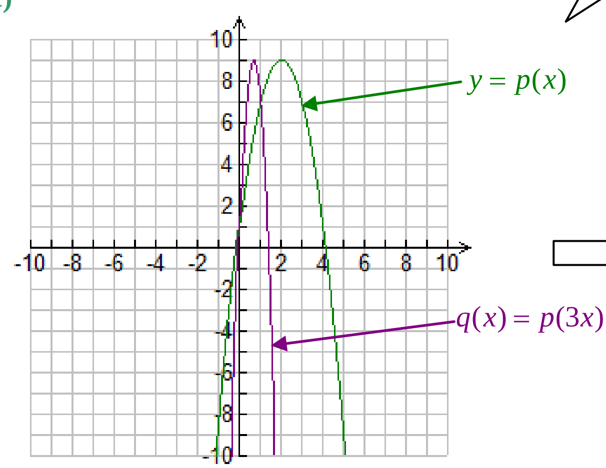
(b)



$(x,y)\to(x,y+3)$

e.g. $(2,6)\to(2,6+3)=(2,9)$

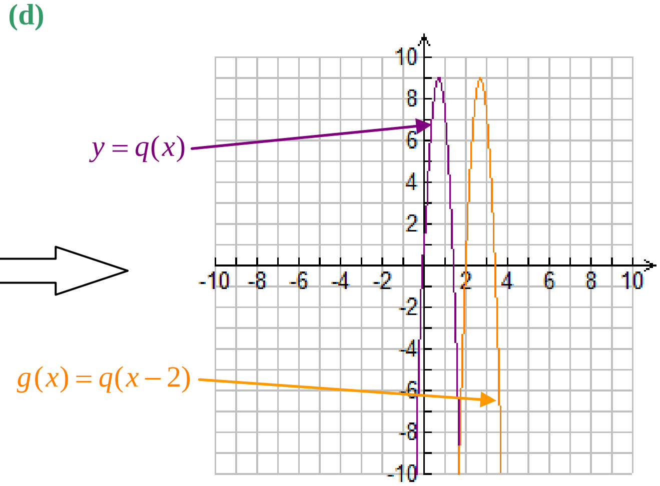
(c)



$(x,y)\to(\frac{1}{3}x,y)$

e.g. $(2,9)\to\left(\frac{1}{3}(2),9\right)=\left(\frac{2}{3},9\right)$

(d)



$(x,y)\to(x+2,y)$

e.g. $\left(\frac{2}{3},9\right)\to\left(\frac{2}{3}+2,9\right)=\left(\frac{8}{3},9\right)$

Example 2

(a) Determine the zeros of $f(x) = 12x^2 - 19x + 5$.

(b) Determine the zeros of $g(x) = -3f(-2x-7)$.

Solution

(a) $12x^2 - 19x + 5 = 0$

$\therefore 12x^2 - 4x - 15x + 5 = 0$

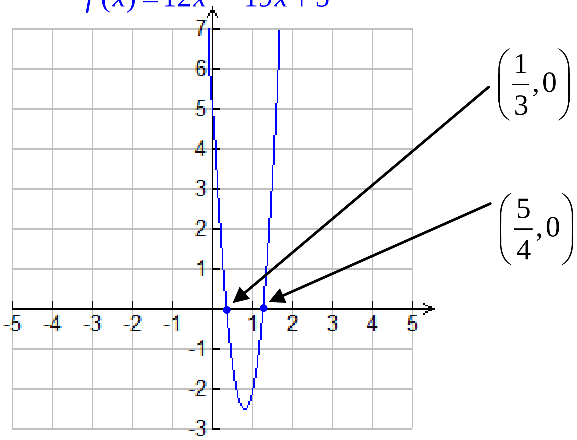
$\therefore 4x(3x-1) - 5(3x-1) = 0$

$\therefore (3x-1)(4x-5) = 0$

$\therefore 3x - 1 = 0$ or $4x - 5 = 0$

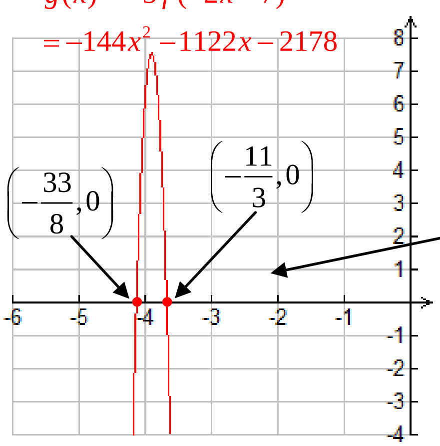
$\therefore x = \frac{1}{3}$ or $x = \frac{5}{4}$

$f(x) = 12x^2 - 19x + 5$



$g(x) = -3f(-2x-7)$

$= -144x^2 - 1122x - 2178$



(b) ***Algebraic Solution***

The zeros of g can be found by solving the equation $g(x) = 0$, that is, $-3f(-2x-7) = 0$. To this end, it is easier to use the factored form of f, $f(x) = (3x-1)(4x-5)$

$g(x) = 0$

$\therefore -3f(-2x-7) = 0$

$\therefore -3[3(-2x-7)-1][4(-2x-7)-5] = 0$

$\therefore (-6x-21-1)(-8x-28-5) = 0$

$\therefore (-6x-22)(-8x-33) = 0$

$\therefore -6x-22 = 0$ or $-8x-33 = 0$

$\therefore x = -\frac{22}{6} = -\frac{11}{3}$ or $x = -\frac{33}{8}$

> Recall that the horizontal transformations are done in the ***reverse*** order. This is why the image of x is $-\frac{1}{2}(x+7)$, which corresponds to a shift 7 units right followed by a compression by a factor of $-\frac{1}{2}$.

Geometric Solution

Since the given transformation does not involve a vertical shift, the zeros of g are simply the images of the zeros of f under the given transformation.

Pre-image	***Image***
(x, y)	$\left(-\frac{1}{2}(x+7), -3y\right)$
$\left(\frac{1}{3}, 0\right)$	$\left(-\frac{1}{2}(\frac{1}{3}+7), -3(0)\right) = \left(-\frac{11}{3}, 0\right)$
$\left(\frac{5}{4}, 0\right)$	$\left(-\frac{1}{2}(\frac{5}{4}+7), -3(0)\right) = \left(-\frac{33}{8}, 0\right)$

Notice that the roots of $g(x) = 0$ are the ***images*** of the roots of $f(x) = 0$.

Homework

Read pp. 233 – 239

pp. 240 – 243, 1bef, 3cfh, 4bh, 5ef, 6adf, 7bdfh, 8f, 9df, 14, 15, 16, 17, 18, 19, 21

Inverses of Functions

Introduction – The Notion of an Inverse

On an intuitive level, the ***inverse of a function*** is simply its ***opposite***. The following table lists some common operations and their opposites.

Some Operations and their Inverses		*Example*		*Observations*		*Conclusion*
$+$	$-$	$\mathbf{3}+5=8$	$8-5=\mathbf{3}$	$+:\mathbf{3}\to 8$	$-:8\to\mathbf{3}$	The ***inverse of an operation*** "gets you back to where you started." It ***undoes the operation***.
$\times$	$\div$	$\mathbf{3}\times 5=15$	$15\div 5=\mathbf{3}$	$\times:\mathbf{3}\to 15$	$\div:15\to\mathbf{3}$	
square a number	square root	$\mathbf{5}^2=25$	$\sqrt{25}=\mathbf{5}$	$^2:\mathbf{5}\to 25$	$\sqrt{\ }:25\to\mathbf{5}$	

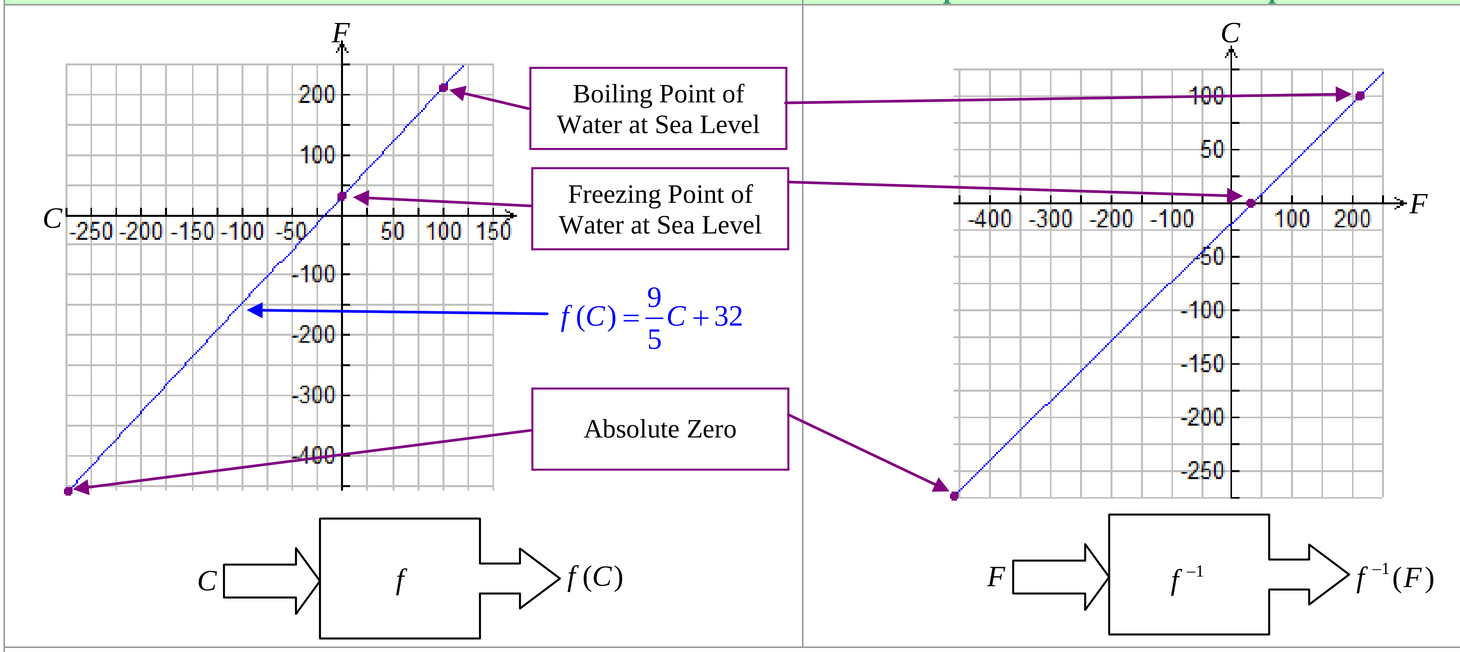
A Classic Example of a Function and its Inverse

With the exception of the United States and perhaps a very small number of other countries, the Celsius scale is used to measure temperature for weather forecasts and many other purposes. In the United States, however, the Fahrenheit scale is still used for most non-scientific purposes. The following shows you how the two scales are related.

C = degrees Celsius, F = degrees Fahrenheit

f = function that "outputs" the Fahrenheit temperature when given the Celsius temperature C as input

A Function f that Converts Celsius Temperatures into Fahrenheit Temperatures	*The Inverse of f, written f^{-1}, converts Fahrenheit Temperatures into Celsius Temperatures*



To obtain the equation of f^{-1}, apply the inverse operations in the reverse order:

Forward: $C\to\times(9/5)\to+32\to F$ ***Reverse:*** $F\to-32\to\div(9/5)\to C$ $\therefore C=f^{-1}(F)=\frac{5}{9}(F-32)$

(Note that dividing by 9/5 is the same as multiplying by 5/9.)

Understanding the Inverse of a Function from a Variety of Perspectives

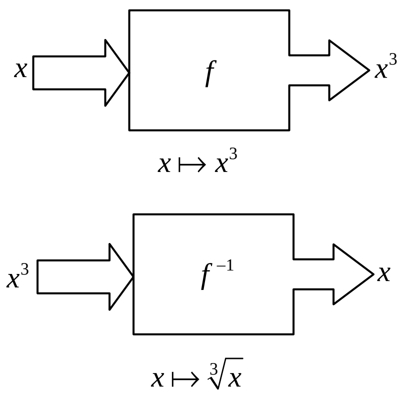
- It is critical that you understand that the ***inverse*** of a function is its ***opposite***. That is, the inverse of a function must ***undo*** whatever the function does.
- The inverse of a function is denoted f^{-1}. It is important to comprehend that the "–1" in this notation is ***not an exponent***. The symbol f^{-1} means "the inverse of the function *f*," ***not*** $\frac{1}{f}$.
- The notation $x\mapsto f(x)$, called ***mapping notation***, can be used to convey the same idea as a function machine.

Example 1

Does $f(x)=x^3$ have an inverse? If so, what is the inverse function of $f(x)=x^3$?

Solution

By examining the ***six different perspectives of functions*** that we have considered throughout this unit, we can easily convince ourselves that $f(x)=x^3$ does have an inverse, namely $f^{-1}(x)=\sqrt[3]{x}$.



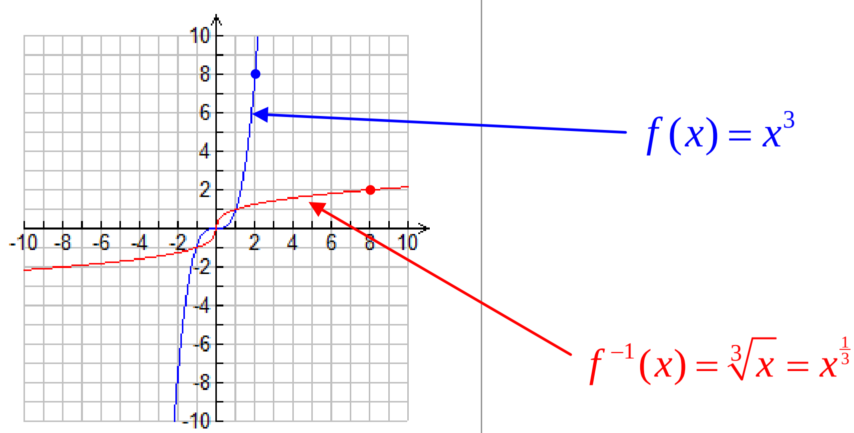
$f=\{(-3,-27),(-2,-8),(2,8),(3,27),\ldots\}$

$f^{-1}=\{(-27,-3),(-8,-2),(8,2),(27,3),\ldots\}$

Note that the ordered pairs of f^{-1} are "***reversed***." That is, the *x* and *y*-co-ordinates of the ordered pairs of *f* are interchanged to obtain the ordered pairs of f^{-1}.

x	$f(x)$	x	$f^{-1}(x)$
–3	–27	–27	–3
–2	–8	–8	–2
–1	–1	–1	–1
0	0	0	0
1	1	1	1
2	8	8	2
3	27	27	3
.	.	.	.
.	.	.	.
.	.	.	.

–3		–27		–27		–3
–2	→	–8		–8	→	–2
–1	→	–1		–1	→	–1
0	→	0		0	→	0
1	→	1		1	→	1
2	→	8		8	→	2
3	→	27		27	→	3
.	→	.		.	→	.
.		.		.		.
.		.		.		.



Example 2

Does $f(x)=x^2$ have an inverse? If so, what is the inverse function of $f(x)=x^2$?

Solution

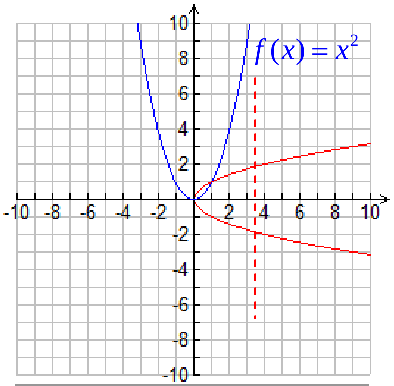
In the last example we learned that the inverse of a function *f* is obtained by interchanging the *x* and *y*-co-ordinates of the ordered pairs of *f*. Let's try this on a few of the ordered pairs of the function $f(x)=x^2$.

$$f=\{(x,y):x\in\mathbb{R},y\in\mathbb{R},y=x^2\}=\{\ldots,(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9),\ldots\}$$

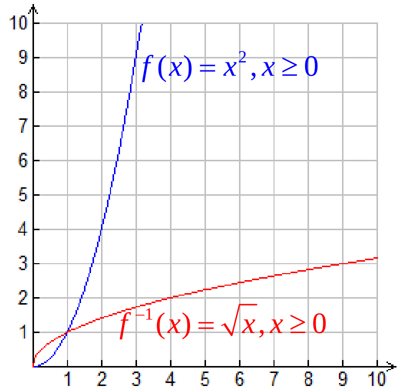
The inverse of *f* ***should be*** the following relation:

$$\{(x,y):(y,x)\in f\}=\{\ldots,(9,-3),(4-2),(1,-1),(0,0),(1,1),(4,2),(9,3),\ldots\}$$

It is readily apparent that there is something wrong, however. This relation is ***not*** a function. Therefore, $f(x)=x^2$ ***does not*** have an inverse function ***unless we restrict its domain***.

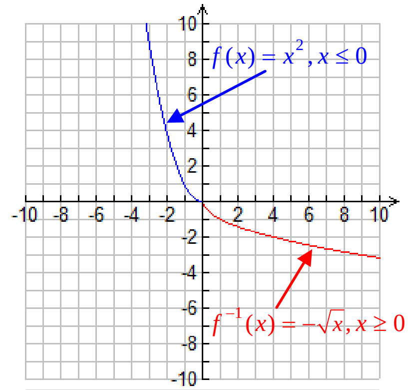


The relation obtained by interchanging the *x* and *y* co-ordinates fails the vertical line test. It is ***not*** a function.



The inverse of *f* is a function if the domain of *f* is restricted to the set of all real numbers $x\ge 0$.

Clearly, $f^{-1}(x)=\sqrt{x}$.



Alternatively, the domain of *f* can be restricted to the set of all real numbers $x\le 0$. In this case, the inverse is $f^{-1}(x)=-\sqrt{x}$.

Observations

1. $f(x)=x^3$ is ***one-to-one*** and has inverse function $f^{-1}(x)=\sqrt[3]{x}$
2. $f(x)=x^2$ is ***many-to-one***; the inverse of *f* is not a function unless its domain is restricted to a "piece" of *f* that is one-to-one (e.g. $x\ge 0$ or $x\le 0$)

Summary

We can extend the results of the above examples to all functions.

1. The inverse function f^{-1} of a function *f* exists ***if and only*** *f* is ***one-to-one***. (Technically, *f* must be a bijection. For our purposes, however, it will suffice to require that *f* be one-to-one.)

2. The inverse relation of a ***many-to-one*** function ***is not a function***. However, if the domain of a many-to-one function is restricted in such a way that it is one-to-one for a certain set of "*x*-values," then the inverse relation defined for this "piece" ***is a*** function.

Important Question: How are the Domain and Range of a Function related to the Domain and Range of its Inverse?

Write your answer to this question in the following space.

Important Exercise

Complete the following table. The first two rows are done for you.

Hint: To find the inverse of each given function, apply the ***inverse operations*** in the ***reverse order***.

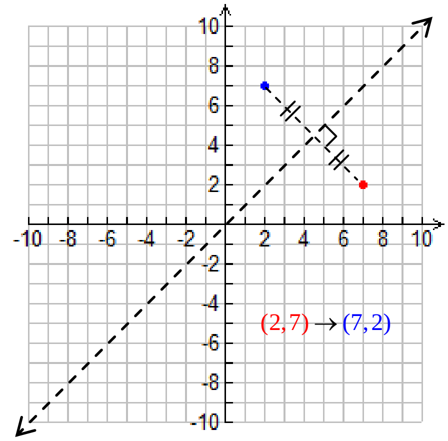
Function: Function Notation	Function: Mapping Notation	One-to-One or Many-to-One?	Inverse Function – State any Restrictions to Domain of f: Function Notation	Inverse Function: Mapping Notation	Domain and Range of f	Domain and Range of f^{-1}
$f(x)=x^3$	$x \mapsto x^3$	one-to-one	$f^{-1}(x)=\sqrt[3]{x}$	$x \mapsto \sqrt[3]{x}$	$D=\mathbb{R}$ $R=\mathbb{R}$	$D=\mathbb{R}$ $R=\mathbb{R}$
$f(x)=x^2$	$x \mapsto x^2$	many-to-one	$f^{-1}(x)=\sqrt{x}$, provided that $x \geq 0$	$x \mapsto \sqrt{x}$	$D=\{x \in \mathbb{R} : x \geq 0\}$ $R=\{y \in \mathbb{R} : y \geq 0\}$	$D=\{x \in \mathbb{R} : x \geq 0\}$ $R=\{y \in \mathbb{R} : y \geq 0\}$
$f(x)=2x+1$						
$f(x)=2x^3-7$						
$f(x)=\frac{1}{x}$						
$f(x)=x^2-4$						
$f(x)=x^2+10x+1$ (This one is a bit tricky. Look ahead to page 40 if you need help.)						

Geometric View of the Inverse of a Function

By now it should be clear that finding the inverse of a function is equivalent to applying the transformation

$$(x, y) \to (y, x)$$

The geometric effect of this transformation can be seen quite easily from the diagram at the right. It is nothing more than a ***reflection in the line y = x***.



> To obtain the graph of $y = f^{-1}(x)$, reflect the graph of $y = f(x)$ in the line $y = x$.

Example

Consider the function $f(x) = x^2 - 5x + 6$.

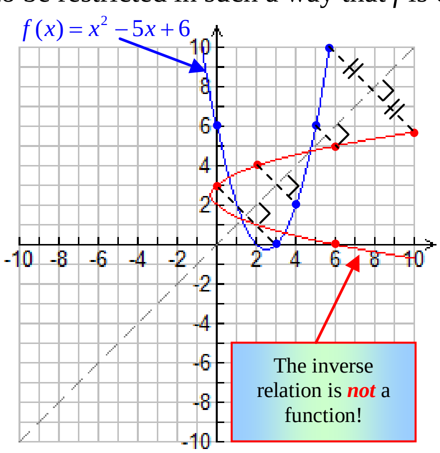
(a) Sketch the graphs of both *f* and f^{-1}.

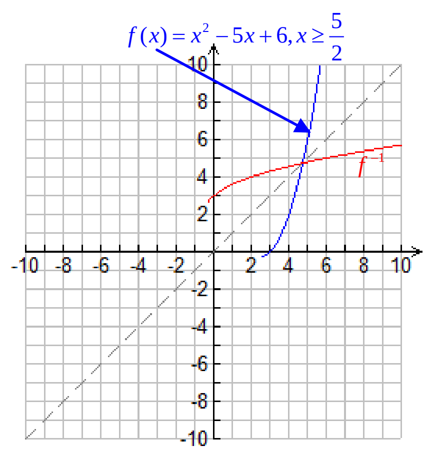
(b) Find the inverse of *f*. State any necessary restrictions to the domain of *f*.

(c) State the domain and range of both *f* and f^{-1}.

Solution

(a) When $f(x) = x^2 - 5x + 6$ is reflected in the line *y* = *x*, the resulting relation is not a function. Therefore, the domain of *f* needs to be restricted in such a way that *f* is one-to-one for the restricted set of *x*-values.

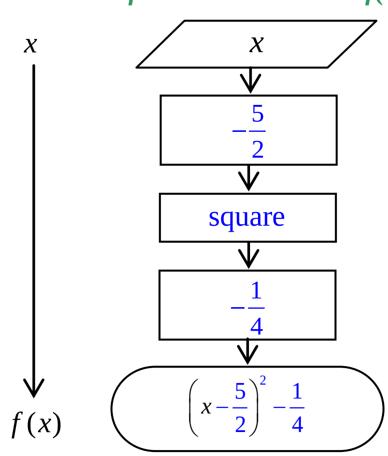




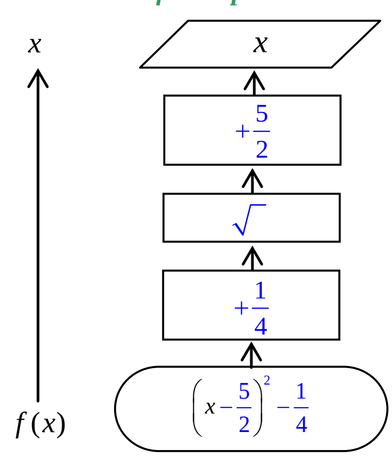
(b) Since *x* appears in two terms, it is not possible to apply the inverse operations in the reverse order directly. Once again then, our good friend "completing the square" comes to our rescue.

$$f(x) = x^2 - 5x + 6$$
$$= x^2 - 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 6$$
$$= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{24}{4}$$
$$= \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$$
$$\therefore f(x) = x^2 - 5x + 6 = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}$$

Operations Performed to x to obtain f(x)



Reversal of the Operations



Therefore, to go from *x* to $f^{-1}(x)$ ($x \mapsto f^{-1}(x)$ in mapping notation), we must do the following:

$$x \to +1/4 \to \sqrt{\ } \to +5/2 \to f^{-1}(x)$$

Thus, $f^{-1}(x) = \sqrt{x + \frac{1}{4}} + \frac{5}{2}$.

Alternative Method for Finding $f^{-1}(x)$

1. Apply the transformation $(x, y) \to (y, x)$ (reflection in the line $y = x$). Algebraically this involves ***interchanging*** the x and y variables.
2. Solve for y in terms of x.

$f(x) = x^2 - 5x + 6$

$\therefore y = x^2 - 5x + 6$

Now interchange x and y and solve for y.

$\therefore y^2 - 5y + 6 - x = 0$

Now we can apply the quadratic formula with $a = 1$, $b = -5$ and $c = 6 - x$.

$$\therefore y = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6-x)}}{2(1)}$$

$$\therefore y = \frac{5 \pm \sqrt{25 - 24 + 4x}}{2}$$

$$\therefore y = \frac{5 \pm \sqrt{4x+1}}{2}$$

$$\therefore y = \frac{5}{2} \pm \frac{\sqrt{4x+1}}{2}$$

$$\therefore y = \frac{5}{2} \pm \frac{1}{2}\sqrt{4x+1}$$

$$\therefore y = \frac{5}{2} \pm \frac{1}{2}\sqrt{4\left(x + \frac{1}{4}\right)}$$

$$\therefore y = \frac{5}{2} \pm \frac{1}{2}\sqrt{4}\sqrt{x + \frac{1}{4}}$$

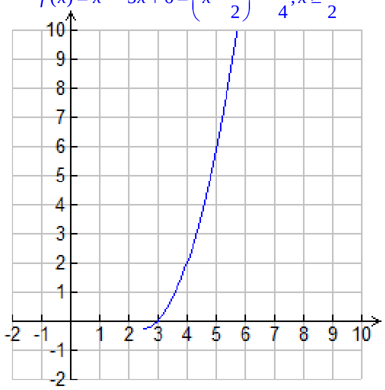
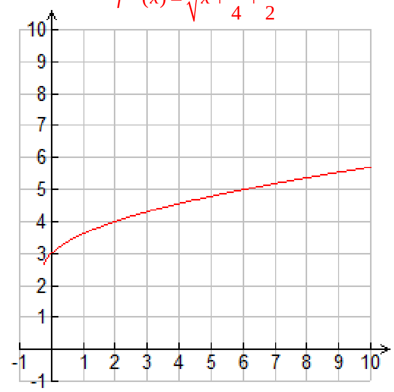
$$\therefore y = \frac{5}{2} \pm \frac{1}{2}(2)\sqrt{x + \frac{1}{4}}$$

$$\therefore y = \frac{5}{2} \pm \sqrt{x + \frac{1}{4}}$$

If we restrict the domain of f to $D = \left\{ x \in \mathbb{R} : x \geq \frac{5}{2} \right\}$, then clearly we should choose $y = \frac{5}{2} + \sqrt{x + \frac{1}{4}} = \sqrt{x + \frac{1}{4}} + \frac{5}{2}$, which agrees with the answer that we obtained using the method of reversing the operations.

Therefore, $f^{-1}(x) = \sqrt{x + \frac{1}{4}} + \frac{5}{2}$

(c) The domain and range of f and f^{-1} can be determined by using a combination of geometry and algebra.

Domain and Range of f	*Domain and Range of f^{-1}*
 From the graph it is clear that the values of x and y need to be restricted. To ensure that f be one-to-one, we were forced to impose the restriction $x \geq 5/2$. By examining the equation of f, it is clear that $-1/4$ is the lowest possible value for $f(x)$. Therefore, $D = \{x \in \mathbb{R} : x \geq 5/2\}$ and $R = \{y \in \mathbb{R} : y \geq -1/4\}$.	 Clearly, the domain and range will be the opposite of the domain and range of f. Therefore, $D = \{x \in \mathbb{R} : x \geq -1/4\}$ and $R = \{y \in \mathbb{R} : y \geq 5/2\}$.

Extremely Important Follow-up Questions

1. If $f(x) = x^3$, we have discovered that $f^{-1}(x) = \sqrt[3]{x}$. Evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$.
2. For any function f, evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$. (**Hint:** It does not matter what f is. All that matters is that f^{-1} is defined at x and at $f(x)$. In addition, keep in mind that an inverse of a function ***undoes*** the function.)
3. The slope of $f(x) = mx + b$ is m. What is the slope of f^{-1}?

Homework
pp. 215 – 220, 2, 3bdfi, 5h, 6f, 7e,f, 8, 9, 11, 14v, 15c, 16viii, 17, 20, 21, 22, 24, 27, 33, 34, 36, 38, 39

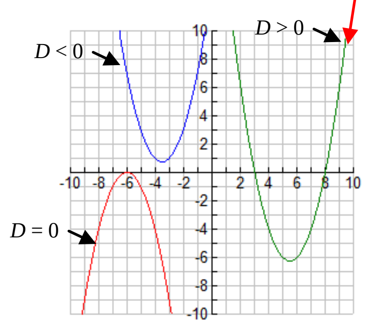
Applications of Quadratic Functions and Transformations

Introduction – Prerequisite Knowledge

The following table summarizes the critically important core knowledge that you must have at your fingertips if you hope to be able to solve a significant percentage of the problems in this section.

Relations and Functions

- ***Idea of Mathematical Relationship***
- ***Definition of Relation***
- ***Definition of Function***
- ***Six Perspectives***
  Set of ordered pairs, function machine, table of values, mapping diagram, graphical (geometric), algebraic
- ***Discrete and Continuous Relations***
- ***Function Notation***
  $x \to$ independent variable
  $f(x) \to$ dependent variable
- ***Mapping Notation***
  "input" $\mapsto$ "output"
  $x \mapsto f(x)$
- ***Domain and Range***



Quadratic Functions

- ***Solving Quadratic Equations***
  First write the quadratic equation in the form $ax^2+bx+c=0$.
  Try ***factoring first***.
  If the quadratic does not factor, use the ***quadratic formula***.
  Only use the method of "completing the square" if you are asked to!
  The ***nature of the roots*** can be determined by calculating the ***discriminant*** $D=b^2-4ac$.
- ***General Form of Quadratic Function***
  $f(x)=ax^2+bx+c$
- ***Vertex Form of Quadratic Function***
  $f(x)=a(x-h)^2+k$
- ***Rate of Change of Quadratic Functions***
  Quadratic functions are used to model quantities whose rate of change is linear. That is, the ***first differences change linearly*** and ***the second differences are constant***.

Transformations

- ***Transformations Studied***
  Horizontal and Vertical
  - Translations (Shifts)
  - Reflections in Axes
  - Stretches & Compressions

  A reflection in the *x*-axis is the same as a vertical stretch by –1
  A reflection in the *y*-axis is the same as a horizontal stretch by –1
- ***Mapping Notation***
  Horizontal and Vertical Translations
  $(x,y)\to(x+h,y+k)$
  Reflections in Vertical and Horizontal Axes
  $(x,y)\to(-x,-y)$
  Horizontal and Vertical Stretches and Compressions
  $(x,y)\to(ax,by)$
  Combination of Stretches and Translations
  $(x,y)\to(ax+h,by+k)$
- ***Equivalent Transformations in Function Notation***
  Horizontal and Vertical Translations
  $g(x)=f(x-h)+k$
  Reflections in Vertical and Horizontal Axes
  $g(x)=-f(-x)$
  Horizontal and Vertical Stretches and Compressions
  $g(x)=af(b^{-1}x)=af((1/b)x)$
  Combination of Stretches and Translations
  $g(x)=af(b^{-1}(x-h))+k$

Functions and their Inverses

- The inverse of a relation is found by applying the transformation $(x,y)\to(y,x)$, which is a reflection in the line $y=x$.
- If *f* is a function and its inverse is a function, the inverse function is denoted f^{-1}.
- The inverse of a function undoes the function. That is, $f(f^{-1}(x))=f^{-1}(f(x))=x$.
- The inverse of a function can be found by applying the inverse operations in the reverse order.
- The inverse of a function can also be found by applying the transformation $(x,y)\to(y,x)$ (interchanging *x* and *y*) and then solving for *y* in terms of *x*.
- A one-to-one function always has an inverse function.
- A many-to-one function, on the other hand, must have its domain restricted in order for an inverse function to be defined.

Problem Solving Activity

You are to do your best to try to solve each of the following problems without assistance. Although answers can be found in the "Activity Solutions" document for unit 1, ***you should not consult them until you have made a concerted, independent effort to develop your own solutions!***

1. The word "function" is often used to express relationships that are not mathematical. Explain the meaning of the following statements:

(a) Crime is a function of socioeconomic status.

(b) Financial independence is a function of education and hard work.

(c) Success in school is a function of students' work habits, quality of teaching, parental support and intelligence.

2. Investigate, both graphically and algebraically, the transformations that affect the number of roots of the following quadratic equations.

(a) $x^2 = 0$ **(b)** $-x^2 + 4 = 0$ **(c)** $-x^2 + x + 56 = 0$ **(d)** $3x^2 - 16x - 35 = 0$

3. Although the method of completing the square is very powerful, it can involve a great deal of work. If your goal is to find the maximum or minimum of a quadratic function, you can use a simpler method called ***partial factoring***. Explain how partially factoring $f(x) = 3x^2 - 6x + 5$ into the form $f(x) = 3x(x-2) + 5$ helps you determine the minimum of the function.

4. Find the ***maximum or minimum*** value of each quadratic function using ***three different algebraic methods***. Check your answers by using a graphical (geometric) method.

(a) $h(t) = -4.9t^2 + 4.9t + 274.4$ **(b)** $g(y) = 6y^2 - 5y - 25$

5. Given the quadratic function $f(x) = (x-3)^2 - 2$ and $g(x) = af(b(x-h)) + k$, describe the effect of each of the following. If you need assistance, you can use a "slider control" in TI-Interactive (to be demonstrated in class).

(a) $a = 0$ **(b)** $b = 0$ **(c)** $a > 1$ **(d)** $a < -1$

(e) $0 < a < 1$ **(f)** $-1 < a < 0$ **(g)** $b > 1$ **(h)** $b < -1$

(i) $0 < b < 1$ **(j)** $-1 < b < 0$ **(k)** k is increased **(l)** k is decreased

(m) h is increased **(n)** h is decreased **(o)** $a = 1$, $b = 1$, $h = 0$, $k = 0$ **(p)** $a = -1$, $b = -1$, $h = 0$, $k = 0$

6. The profit, $P(x)$, of a video company, in ***thousands of dollars***, is given by $P(x) = -5x^2 + 550x - 5000$, where x is the amount spent on advertising, in thousands of dollars. Determine the maximum profit that the company can make and the amounts spent on advertising that will result in a profit of at least $4000000.

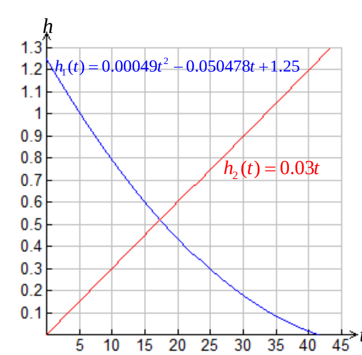
7. Suppose that a quadratic function $f(x) = ax^2 + bx + c$ has x-intercepts r_1 and r_2. Describe transformations of f that produce quadratic functions with the same x-intercepts. That is, describe transformations of f under which the points $(r_1, 0)$ and $(r_2, 0)$ are ***invariant***.

8. Determine the equation of the quadratic function that passes through (2, 5) if the roots of the corresponding quadratic equation are $1-\sqrt{5}$ and $1+\sqrt{5}$.

9. Determine, through investigation, the equations of the lines that have a slope of 2 and that intersect the quadratic function $f(x) = x(6-x)$

(a) once **(b)** twice **(c)** never

10. At a wild party, some inquisitive MCR3U0 students performed an interesting experiment. They obtained two containers, one of which was filled to the brim with a popular party beverage while the other was empty. The full container was placed on a table and the empty container was placed on the floor right next to it. Then, a hole was poked near the bottom of the first container, which caused the party "liquid" to drain out of the first container and into the other. By carefully collecting and analyzing data, the students determined two functions that modelled how the heights of the liquids (in metres) varied with time (in seconds): $h_1(t) = 0.00049t^2 - 0.050478t + 1.25$ and $h_2(t) = 0.03t$

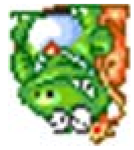
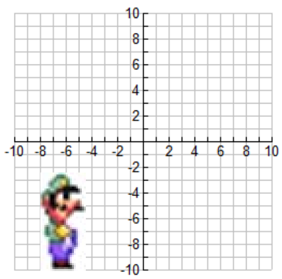
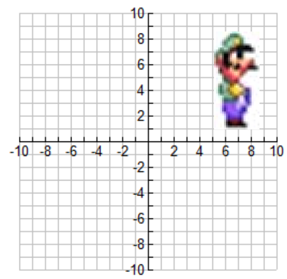


(a) Which function applies to the container with the hole? Explain.

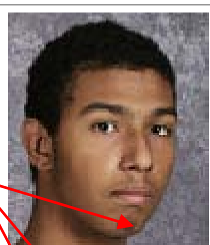
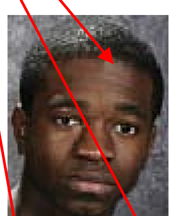
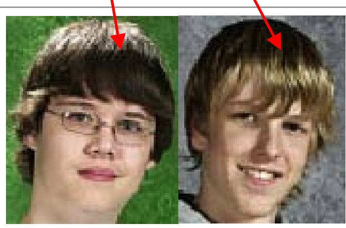
(b) At what time were the heights of the liquids equal?

(c) Explain why the height of the liquid in one container varied linearly with time while the height in the other container varied quadratically with time.

11. Video games depend heavily on transformations. The following table gives a few examples of simple transformations you might see while playing Super Mario Bros. Complete the table.

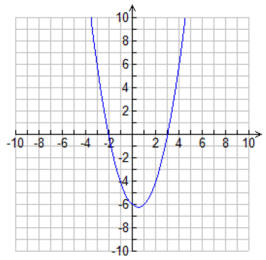
Before Transformation	*After Transformation*	*Nature of the Transformation*	*Use in the Video Game*
		Reflection in the vertical line that divides Mario's body in half	
			
			

12. Complete the following table.

Before Transformation	*After Transformation*	*Nature of the Transformation*
		
		
		
		

Found Guilty of being Physically Present but Mentally Absent!

13. For each quadratic function, determine the number of zeros (*x*-intercepts) using the methods listed in the table.

Quadratic Function	***Graph***	***Discriminant*** $D = b^2 - 4ac$	***Factored Form***	***Conclusion – Number of Zeros***
$f(x) = x^2 - x - 6$		$a = 1, b = -1, c = -6$ $\therefore D = b^2 - 4ac$ $= (-1)^2 - 4(1)(-6)$ $= 1 + 24$ $= 25$	$x^2 - x - 6 = 0$ $\therefore (x-3)(x+2) = 0$ $\therefore x - 3 = 0$ or $x + 2 = 0$ $\therefore x = 3$ or $x = -2$	• Since there are 2 *x*-intercepts, *f* has two zeros. • Since $D > 0$, *f* has two zeros. • Since $x^2 - x - 6 = 0$ has two solutions, *f* has two zeros.
$g(t) = t^2 - t + 6$				
$P(t) = 12t^2 - 19t - 5$				
$q(z) = 9z^2 - 6z + 1$				
$f(x) = 49x^2 - 100$				

14. Use the quadratic formula to explain how the discriminant allows us to predict the number of roots of a quadratic equation.

15. How can the discriminant be used to predict whether a quadratic function can be factored? Explain.

16. The following table lists the approximate accelerations due to gravity near the surface of the Earth, moon and sun.

Earth	*Jupiter*	*Saturn*
9.87 m/s^2	25.95 m/s^2	11.08 m/s^2

The data in the above table lead to the following equations for the height of an object dropped near the surface of each of the celestial bodies given above. In each case, $h(t)$ represents the height, in metres, of an object above the surface of the body t seconds after it is dropped from an initial height h_0.

Earth	*Jupiter*	*Saturn*
$h(t) = -4.94t^2 + h_0$	$h(t) = -12.97t^2 + h_0$	$h(t) = -5.54t^2 + h_0$

In questions (a) to (d), use an initial height of 100 m for the Earth, 200 m for Saturn and 300 m for Jupiter.

(a) On the same grid, sketch each function.

(b) Explain how the "Jupiter function" can be transformed into the "Saturn function."

(c) Consider the graphs for Jupiter and Saturn. Explain the ***physical meaning*** of the point(s) of intersection of the two graphs.

(d) State the domain and range of each function. Keep in mind that each function is used to ***model*** a physical situation, which means that the allowable values of t are restricted.

17. Have you ever wondered why an object that is thrown up into the air always falls back to the ground? Essentially, this happens because the kinetic energy of the object (energy of motion) is less than its potential energy (the energy that the Earth's gravitational field imparts to the object). As long as an object's kinetic energy is less than its potential energy, it will either fall back to the ground or remain bound in a closed orbit around the Earth. On the other hand, if an object's kinetic energy exceeds its potential energy, then it can break free from the Earth's gravitational field and escape into space.

The function E_Δ, defined by the equation $E_\Delta(v) = \dfrac{mv^2}{2} - \dfrac{GMm}{r} = m\left(\dfrac{v^2}{2} - \dfrac{GM}{r}\right)$, gives the difference between the kinetic energy and potential energy of an object of mass m moving with velocity v in the gravitational field of a body of mass M and radius r. In addition, G represents the universal gravitational constant and is equal to 6.67429×10^{-11} m^3 kg^{-1} s^{-2}.

(a) Use the data in the table to determine E_Δ for the Earth, Jupiter and Saturn for an object with a mass of 1 kg.

Planet	*Radius (m)*	*Mass (kg)*	E_Δ
Earth	6.38×10^6	5.98×10^{24}	
Jupiter	7.15×10^7	1.90×10^{27}	
Saturn	6.03×10^7	5.68×10^{26}	

(b) Sketch the graph of E_Δ for the Earth, Jupiter and Saturn (for an object of a mass of 1 kg).

(c) An object can escape a body's gravitational field if its kinetic energy exceeds its potential energy. Using the graphs from part (b), determine the escape velocity for each of the given planets.

(d) Does the escape velocity of an object depend on its mass?

18. Suppose that $f(x) = x^2 - 5x$ and that $g(x) = 2f^{-1}(\frac{1}{3}x - 3) + 1 = 2f^{-1}(\frac{1}{3}(x-9)) + 1$.

(a) The following table lists the transformations, in mapping notation, applied to *f* to obtain *g*. Give a verbal description of each transformation.

	Mapping Notation	*Verbal Description*
Vertical	$(x, y) \to (x, 2y+1)$	
Horizontal	$(x, y) \to (3x+9, y)$	
Other	$(x, y) \to (y, x)$	

(b) As you already know, horizontal and vertical transformations are independent of each other, which means that the results are the same regardless of the order in which they are applied. Does it matter at what point in the process the transformation $(x, y) \to (y, x)$ is applied? Explain.

(c) Sketch the graph of *g*.

(d) State an equation of *g*.

Summary of Problem Solving Strategies used in this Section

Problem	*Strategies*
Determine the number of zeros of a quadratic function. (This is equivalent to finding the number of *x*-intercepts.)	**1.**
	2.
	3.
Predict whether a quadratic function can be factored.	
Find the maximum or minimum value of a quadratic function. (This is equivalent to finding the vertex of the corresponding parabola.)	**1.**
	2.
	3.
Find the equation of a quadratic function that passes through a given point and whose *x*-intercepts are the same as the roots of a corresponding quadratic equation.	
Intersection of a Linear Function and a Quadratic Function	

OVERALL EXPECTATIONS

By the end of this course, students will:

1. demonstrate an understanding of functions, their representations, and their inverses, and make connections between the algebraic and graphical representations of functions using transformations;
2. determine the zeros and the maximum or minimum of a quadratic function, and solve problems involving quadratic functions, including problems arising from real-world applications;
3. demonstrate an understanding of equivalence as it relates to simplifying polynomial, radical, and rational expressions.

1. Representing Functions

By the end of this course, students will:

1.1 explain the meaning of the term *function*, and distinguish a function from a relation that is not a function, through investigation of linear and quadratic relations using a variety of representations (i.e., tables of values, mapping diagrams, graphs, function machines, equations) and strategies (e.g., identifying a one-to-one or many-to-one mapping; using the vertical-line test)

Sample problem: Investigate, using numeric and graphical representations, whether the relation $x = y^2$ is a function, and justify your reasoning.

1.2 represent linear and quadratic functions using function notation, given their equations, tables of values, or graphs, and substitute into and evaluate functions [e.g., evaluate $f\left(\frac{1}{2}\right)$, given $f(x) = 2x^2 + 3x - 1$]

1.3 explain the meanings of the terms *domain* and *range*, through investigation using numeric, graphical, and algebraic representations of the functions $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$; describe the domain and range of a function appropriately (e.g., for $y = x^2 + 1$, the domain is the set of real numbers, and the range is $y \geq 1$); and explain any restrictions on the domain and range in contexts arising from real-world applications

Sample problem: A quadratic function represents the relationship between the height of a ball and the time elapsed since the ball was thrown. What physical factors will restrict the domain and range of the quadratic function?

1.4 relate the process of determining the inverse of a function to their understanding of reverse processes (e.g., applying inverse operations)

1.5 determine the numeric or graphical representation of the inverse of a linear or quadratic function, given the numeric, graphical, or algebraic representation of the function, and make connections, through investigation using a variety of tools (e.g., graphing technology, Mira, tracing paper), between the graph of a function and the graph of its inverse (e.g., the graph of the inverse is the reflection of the graph of the function in the line $y = x$)

Sample problem: Given a graph and a table of values representing population over time, produce a table of values for the inverse and graph the inverse on a new set of axes.

1.6 determine, through investigation, the relationship between the domain and range of a function and the domain and range of the inverse relation, and determine whether or not the inverse relation is a function

Sample problem: Given the graph of $f(x) = x^2$, graph the inverse relation. Compare the domain and range of the function with the domain and range of the inverse relation, and investigate connections to the domain and range of the functions $g(x) = \sqrt{x}$ and $h(x) = -\sqrt{x}$.

1.7 determine, using function notation when appropriate, the algebraic representation of the inverse of a linear or quadratic function, given the algebraic representation of the function [e.g., $f(x) = (x - 2)^2 - 5$], and make connections, through investigation using a variety of tools (e.g., graphing technology, Mira, tracing paper), between the algebraic representations of a function and its inverse (e.g., the inverse of a linear function involves applying the inverse operations in the reverse order)

Sample problem: Given the equations of several linear functions, graph the functions and their inverses, determine the equations of the inverses, and look for patterns that connect the equation of each linear function with the equation of the inverse.

1.8 determine, through investigation using technology, the roles of the parameters a, k, d, and c in functions of the form $y = af(k(x - d)) + c$, and describe these roles in terms of transformations on the graphs of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$ (i.e., translations; reflections in the axes; vertical and horizontal stretches and compressions to and from the x- and y-axes)

Sample problem: Investigate the graph $f(x) = 3(x - d)^2 + 5$ for various values of d, using technology, and describe the effects of changing d in terms of a transformation.

1.9 sketch graphs of $y = af(k(x - d)) + c$ by applying one or more transformations to the graphs of $f(x) = x$, $f(x) = x^2$, $f(x) = \sqrt{x}$, and $f(x) = \frac{1}{x}$, and state the domain and range of the transformed functions

Sample problem: Transform the graph of $f(x)$ to sketch $g(x)$, and state the domain and range of each function, for the following: $f(x) = \sqrt{x}$, $g(x) = \sqrt{x - 4}$; $f(x) = \frac{1}{x}$, $g(x) = -\frac{1}{x + 1}$.

2. Solving Problems Involving Quadratic Functions

By the end of this course, students will:

2.1 determine the number of zeros (i.e., x-intercepts) of a quadratic function, using a variety of strategies (e.g., inspecting graphs; factoring; calculating the discriminant)

Sample problem: Investigate, using graphing technology and algebraic techniques, the transformations that affect the number of zeros for a given quadratic function.

2.2 determine the maximum or minimum value of a quadratic function whose equation is given in the form $f(x) = ax^2 + bx + c$, using an algebraic method (e.g., completing the square; factoring to determine the zeros and averaging the zeros)

Sample problem: Explain how partially factoring $f(x) = 3x^2 - 6x + 5$ into the form $f(x) = 3x(x - 2) + 5$ helps you determine the minimum of the function.

2.3 solve problems involving quadratic functions arising from real-world applications and represented using function notation

Sample problem: The profit, $P(x)$, of a video company, in thousands of dollars, is given by $P(x) = -5x^2 + 550x - 5000$, where x is the amount spent on advertising, in thousands of dollars. Determine the maximum profit that the company can make, and the amounts spent on advertising that will result in a profit and that will result in a profit of at least \$4 000 000.

2.4 determine, through investigation, the transformational relationship among the family of quadratic functions that have the same zeros, and determine the algebraic representation of a quadratic function, given the real roots of the corresponding quadratic equation and a point on the function

Sample problem: Determine the equation of the quadratic function that passes through (2, 5) if the roots of the corresponding quadratic equation are $1 + \sqrt{5}$ and $1 - \sqrt{5}$.

2.5 solve problems involving the intersection of a linear function and a quadratic function graphically and algebraically (e.g., determine the time when two identical cylindrical water tanks contain equal volumes of water, if one tank is being filled at a constant rate and the other is being emptied through a hole in the bottom)

Sample problem: Determine, through investigation, the equations of the lines that have a slope of 2 and that intersect the quadratic function $f(x) = x(6 - x)$ once; twice; never.

3. Determining Equivalent Algebraic Expressions*

By the end of this course, students will:

3.1 simplify polynomial expressions by adding, subtracting, and multiplying

Sample problem: Write and simplify an expression for the volume of a cube with edge length $2x + 1$.

3.2 verify, through investigation with and without technology, that $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, $a \geq 0$, $b \geq 0$, and use this relationship to simplify radicals (e.g., $\sqrt{24}$) and radical expressions obtained by adding, subtracting, and multiplying [e.g., $(2 + \sqrt{6})(3 - \sqrt{12})$]

3.3 simplify rational expressions by adding, subtracting, multiplying, and dividing, and state the restrictions on the variable values

Sample problem: Simplify $\dfrac{2x}{4x^2 + 6x} - \dfrac{3}{2x + 3}$, and state the restrictions on the variable.

3.4 determine if two given algebraic expressions are equivalent (i.e., by simplifying; by substituting values)

Sample problem: Determine if the expressions $\dfrac{2x^2 - 4x - 6}{x + 1}$ and $8x^2 - 2x(4x - 1) - 6$ are equivalent.

*The knowledge and skills described in the expectations in this section are to be introduced as needed, and applied and consolidated, as appropriate, in solving problems throughout the course.